Dephasing of a non-relativistic quantum particle due to a conformally fluctuating spacetime

Paolo M. Bonifacio[1], **Charles H.-T. Wang**[1,2], **J. Tito Mendonça**[2,3], **Robert Bingham**[2,4]

[1]SUPA Department of Physics, University of Aberdeen, King's College, Aberdeen AB24 3UE, UK

[2]Rutherford Appleton Laboratory, STFC, Chilton, Didcot, Oxfordshire OX11 0QX, UK

[3]CFP and CFIF, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

[4]SUPA Department of Physics, University of Strathclyde, Glasgow G4 0NG, UK

E-mail: p.bonifacio@abdn.ac.uk, c.wang@abdn.ac.uk, titomend@ist.utl.pt, r.bingham@rl.ac.uk

Abstract. We investigate the dephasing suffered by a nonrelativistic quantum particle within a conformally fluctuating spacetime geometry. Starting from a minimally coupled massive Klein-Gordon field, the low velocity limit yields an effective Schrödinger equation where the wave function couples to gravity through an effective nonlinear potential induced by the conformal fluctuations. The quantum evolution is studied through a Dyson expansion scheme up to second order. We show that only the nonlinear part of the potential can induce dephasing. This happens through an exponential decay of the off diagonal terms of the particle density matrix. The bath of conformal radiation is modeled in 3-dimensions and its statistical properties are described in general in terms of a power spectral density. The case of a Lorentz invariant spectral density, allowing to model vacuum fluctuations at a low energy domain, is investigated and a general formula describing the loss of coherence derived. This depends quadratically on the particle mass and on the inverse cube of a typical particle dependent cutoff scale. Finally, the possibilities for experimental verification are discussed. It is shown that current interferometry experiments cannot detect such an effect. However this conclusion may improve by using high mass entangled quantum states.

PACS numbers: 03.65.-w, 03.65.Yz, 04.20.Cv, 05.40.-a

1. Introduction

It is generally agreed that the underlying quantum nature of gravity implies that the spacetime structure close to the Planck scale departs from that predicted by General Relativity. Unfortunately the Quantum Gravity domain is still beyond modern particle accelerators such as LHC. Nonetheless, finding experimental ways to test the quantum structure of spacetime would be highly beneficial to the theoretical developments of our fundamental theories of nature. In this respect it is has been suggested that quantum gravity could induce decoherence on a quantum particle through its underlying Planck scale spacetime fluctuations [1, 2, 3, 4, 5, 6].

As the sensitivity and performance of matter wave interferometers is increasing [7, 8, 9, 10, 11], it is important to assess the theoretical possibility of a future

experimental detection of intrinsic, spacetime induced decoherence. The closely related dephasing effect due to a random bath of classical GWs (e.g. of astrophysical origin) has been extensively studied e.g. in [12]. The problem of the decoherence induced by spacetime fluctuations is difficult to study as a quantum gravity theory is still missing. Notwithstanding promising progress, mainly in loop quantum gravity and superstring theory [13], a coherent and established quantum gravity theoretical framework is still missing. Thus any theoretical attempt for a prediction of the decoherence induced by spacetime fluctuations must exploit some semiclassical framework. Such approaches typically represent the spacetime metric close to the Planck scale by means of fluctuating functions. These are usually supposed to mimic the vacuum quantum property of spacetime down to some cutoff scale $\ell = \lambda L_P$, where L_P is the Planck scale. The adimensional parameter λ marks the benchmark between the fully quantum regime and the scale where the classical properties of spacetime start to emerge [1]. The fact that classical fluctuating fields can be used to reproduce various genuine quantum effects is well known, e.g. from the work of Boyer [14, 15] in the case of the EM field or Frederick [16] in the case of spacetime fluctuations. This is often exploited in the literature in relation to problems involving the microscopic behavior of the spacetime metric; e.g. a stochastic metric was employed in [17, 18] to study the problem of gravitational collapse and big bang singularities, while in [19] spacetime metric fluctuations were introduced and their ability to induce a WEP violation studied.

A pioneering analysis of the problem of spacetime induced decoherence has been proposed by Power & Percival (PP in the following) [1] in the case of a conformally modulated Minkowski spacetime with conformal fluctuations traveling along 1 space dimension. This was improved by Wang et al. [3], who extended upon PP work attempting to include the effect of GWs. Conformal fluctuations are interesting as they are mathematically easy to treat and offer a convenient way to build 'toy' models to assess some of the problem's features. They have an important role in theoretical physics [20] and are sometimes invoked in the literature also in relation to universal scalar fields [21, 18] that can arise naturally in some modified theories of gravity such as scalar-tensor theories [22, 23].

Within a semiclassical approach that 'replaces' the true quantum environment by classical fluctuating fields we should properly speak of *dephasing* of the quantum particle rather than decoherence. In a remarkable paper [24] about quantum interference in the presence of an environment, Stern and co-authors showed that a fully quantum approach that studies decoherence by 'tracing away' the environment degrees of freedom in the quantum system made up by system + environment, and that in which the dephasing of the quantum particle is due to a stochastic background field give equivalent results.

In this paper we consider a conformally modulated 4-dimensional spacetime metric of the form $g_{ab} = (1 + A)^2\eta_{ab}$. Such a metric has been considered by PP [1], where the dephasing problem was studied in the simple idealized case of a particle propagating in 1-dimension. By imposing Einstein's equation on the metric g_{ab}, PP deduced a wave equation for A. Their procedure to derive an effective newtonian potential interacting with the quantum probe started from the geodesic equation of a test particle. Even though this didn't take properly into account the nonlinearity in the conformal factor $(1 + A)^2$, they found correctly that the change in the density matrix is given by $\delta\rho \propto M^2 T A_0^4 \tau_*$, where M is the probing particle mass, T the flight time, A_0^4 the amplitude of the conformal fluctuations and τ_* their correlation time.

This formula was used to set limits upon λ. However in doing this they did not treat the statistical properties of the fluctuations properly and this resulted in the wrong estimate $\lambda \propto (M^2T/\delta\rho)^{1/7}$, as already noted by Wang et al in [3].

In their work, Wang and co-authors attempt to include GWs into the analysis by considering a metric of the kind $g_{ab} = (1 + A)^2\gamma_{ab}$. This was done by exploiting the results in [25, 26] where a canonical geometrodynamics approach employing a conformal spacial 3-metric was studied. By exploiting an energy density balancing mechanism between the conformal and GWs parts of the total gravitational Hamiltonian the statistical properties of the conformal fluctuations where fixed. This corresponded to assume that each 'quantum' of the conformal field possessed a zero point energy $-\hbar\omega$. Though an improvement over PP work, this approach is still 1-dimensional and too crude to make predictions. Moreover, as it shall be discussed extensively in a future report [27], the issue of energy balance between conformal fluctuations and GWs is a delicate one, and likely not to occur within the standard GR framework.

In the present work we provide a coherent 3-dimensional treatment of the problem of a slow massive test particle coupled to a conformally fluctuating spacetime. The conformal field A is assumed to satisfy a simple wave equation. This will allow a direct comparison with PP result. We also notice that such a framework is expected to arise naturally within a scalar-tensor theory of gravity. This issue will be discussed in a future report [28].

The work is organized as follows: in *section 2* the correct non-relativistic limit of a minimally coupled Klein-Gordon field is deduced and an effective newtonian potential depending nonlinearly on A is identified in the resulting effective Schrödinger equation. In section 3 we set the general formalism to study the average quantum evolution through a Dyson expansion scheme for the particle density matrix ρ. In *section 4*, general results derived in Appendix A are used to model the statistical and correlation properties of the fluctuations through a general, unspecified, power spectral density. In *section 5 and 6* we compute the average quantum evolution and derive a general expression for the evolved density matrix. We show in general that only a nonlinear potential can induce dephasing. The resulting dephasing formula implies an exponential decay of the density matrix off-diagonal elements and is shown to hold in general and independently of the specific spectral properties of the fluctuations. All we assume is that these obey a simple wave equation and that they are a zero mean random process. The overall dephasing predicted within the present 3-dimensional model -equation (28)- is seen to be about two orders of magnitudes larger than in the 1-dimensional case as derived by PP. Next we consider in *section 7* the problem of vacuum fluctuations. To this end a power spectrum $S(\omega) \sim 1/\omega$ is introduced and we derive an explicit formula for the rate of change of the density matrix. This result improves over both Percival's and Wang's work in that its key ingredients are general enough to be potentially suited for a variety of physical situations. Finally the discussion in *section 8* addresses the question of whether the dephasing due to conformal vacuum spacetime fluctuations could be detected. In other words, whether the proposed theory can be falsified or not. A possibility would be through matter wave interferometry employing large molecules. We consider this issue in the final part of this paper by estimating the probing particle resolution scale, setting its ability to be affected by the fluctuations. The resulting formula for the dephasing rate indicates that the level of the effect is still likely to be beyond experimental capability, even for large molecules (e.g. fullerenes) [10]. A measurable effect could possibly result for

larger masses, e.g. if entangles quantum states were employed [29].

2. Low velocity limit and effective Schrödinger equation

The problem we wish to solve is clearly defined: we consider a scalar field A inducing conformal fluctuations on an otherwise flat spacetime geometry according to

$$g_{ab} = (1 + A)^2 \eta_{ab}, \tag{1}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1,)$ is the Minkowski tensor. We will refer to A as to the *conformal field* and this will be assumed to satisfy the wave equation $\partial^c \partial_c A = 0$. Solving this equation with random boundary conditions results in a randomly fluctuating field propagating in 3-dimensional space. We assume this to be a small first order quantity, i.e. $|A| = O(\varepsilon \ll 1)$. Equation (1) expresses the spacetime metric in the laboratory frame. We also suppose that the typical wavelengths of A are effectively cut off at a scale set by $\ell := \lambda L_P$, where $L_P = (\hbar G/c^3)^{1/2} \approx 10^{-35}$ m is the Planck length. The adimensional parameter λ represents a structural property of spacetime marking the quantum-classical transition: below ℓ a full quantum treatment of gravity would be needed so that, by definition, ℓ represents the scale at which a semiclassical approach that treats quantum effects by means of classical randomly fluctuating fields is supposed to be a valid approximation. The value for λ is model dependent but it is generally agreed that $\lambda \gtrsim 10^2$ [3], so that ℓ is expected to be extremely small from a macroscopic point of view. This motivates the assumption that classical macroscopic bodies, including the objects making up the laboratory frame and also the observers, are unaffected by the fluctuations in A. This corresponds to the idea that a physical object is characterized by some typical resolution scale L_{R} that sets its ability to 'feel' the fluctuations: if $L_{\mathrm{R}} \gg \ell$ these average out and do not affect the body, that simply follows the geodesic of the flat background metric. On the other hand a microscopic particle can represent a successful probe of the conformal fluctuations if its resolution scale is small enough.

We are interested in the change of the phase induced on the wave function of a quantum particle by the fluctuating gravitational field. Various approaches to the problem of how spacetime curvature affects the propagation of a quantum wave exist in the literature; e.g. for a stationary, weak field and a non relativistic particle a Schrödinger -like equation can be recovered [30]. The more interesting case of time varying gravitational fields can be treated e.g. by eikonal methods that are usually restricted to weak fields with $g_{ab} = \eta_{ab} + h_{ab}$ and $|h_{ab}| \ll 1$ [31, 32]. Other approaches, e.g. in [19, 6], are based on the scheme developed by Kiefer [33] for the nonrelativistic reduction of a Klein-Gordon field which is minimally coupled to a linearly perturbed metric. The approach of PP in [1] and of Wang at el. in [3] was to derive the geodesic equation in the weak field limit. Their treatments were however employing, incorrectly, the usual newtonian limit scheme which is valid only for weak, linear and static perturbations [34]. Conceptually the wave approach is more satisfying than that based on the geodesic equation because, e.g., the coupling between gravity and a scalar field is well understood and in the appropriate non-relativistic weak field limit an effective Schrödinger equation emerges. This will be our approach below.

We describe the quantum particle of mass M by means of a minimally coupled Klein-Gordon (KG) field ϕ:

$$g^{ab} \nabla_a \nabla_b \phi = \frac{M^2 c^2}{\hbar^2} \phi,$$

where ∇_a is the covariant derivative of the physical metric g_{ab}. Using $g_{ab} = \Omega^2\eta_{ab}$ this equation can easily be made explicit [34] and reads:

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\phi = \frac{\Omega^2 M^2 c^2}{\hbar^2}\phi - 2\partial_a(\ln\Omega)\partial^a\phi, \tag{2}$$

i.e. the wave equation for a massive scalar field plus a perturbation due to A describing the coupling to the conformally fluctuating spacetime. We need to take the appropriate non-relativistic limit in order to deduce an effective Schrödinger equation. Before doing this we remark that, had we consider the alternative meaningful scenario of a conformally coupled scalar field, then the equation $g^{ab}\nabla_a\nabla_b\phi - R\phi/6 - M^2c^2\phi/\hbar^2 = 0$ would read explicitly

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\phi = \frac{\Omega^2 M^2 c^2}{\hbar^2}\phi - 2\partial_a(\ln\Omega)\partial^a\phi - \phi\Omega^{-1}\partial^c\partial_c\Omega.$$

Since it is $\Omega^{-1}\partial^c\partial_c\Omega = (1-A)\partial^c\partial_c A + O(\varepsilon^3)$ we see that *if A is assumed to satisfy the wave equation then the curvature term has no effect*: in this case the minimally and conformally coupled KG equations are equivalent up to second order in A. We also note that, by introducing the *auxiliary field* $\Phi := \Omega\phi$, equation (2) turns out to be equivalent to:

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\Phi = \frac{\Omega^2 M^2 c^2}{\hbar^2}\Phi.$$

In principle, if a solution for Φ were known, then *the physical scalar field representing the particle* would follow formally as $\phi = \Omega^{-1}\Phi = (1-A+A^2)\Phi$, up to second order in A. However in studying the dephasing problem we will *only* find an averaged solution for the average density matrix representing the quantum particle. Therefore, even if a solution in this sense is know in relation to Φ, it would not be obvious how to obtain the corresponding averaged density matrix related to ϕ, which is what we are interested in.

In view of the above considerations we work directly with equation (2) and now proceed in deriving its suitable non-relativistic limit. We will make two assumptions:

(i) the particle is slow i.e., if $\widetilde{p} = M\widetilde{v}$ is its momentum in the laboratory, we have:

$$\frac{\widetilde{v}}{c} \ll 1;$$

(ii) the effect of the conformal fluctuations is small, i.e. the induced change in momentum $\delta p = M\delta v$ is small compared to $M\widetilde{v}$:

$$\frac{\delta v}{\widetilde{v}} \ll 1.$$

In view of these assumptions we can write:

$$\phi = \psi\exp\left(-iMc^2t/\hbar\right),$$

where the field ψ is close to be a plane wave of momentum $\widetilde{p}$. As a consequence we have:

$$\frac{\partial^2\psi}{\partial t^2} \approx -\frac{1}{\hbar^2}\left(\frac{\widetilde{p}^2}{2M}\right)^2\psi.$$

Using this and multiplying by $\hbar^2/2M$, equation (2) yields:

$$\left[\underbrace{i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2M}\nabla^2}_{T_1}-\underbrace{\frac{Mc^2}{8}\left(\frac{\widetilde{v}}{c}\right)^4}_{T_2}\right]\psi =$$

$$=\underbrace{\left(A+\frac{A^2}{2}\right)Mc^2\,\psi}_{T_3}-\underbrace{\frac{\hbar^2}{M}\partial^a\phi\partial_a\ln(1+A)\times\exp\left(iMc^2t/\hbar\right)}_{T_4}. \tag{3}$$

Leaving the term T_4 aside for the moment, the orders of the three underlined terms must be carefully assessed. We have:

$$T_1 \sim M\widetilde{v}^2, \quad T_2 \sim Mc^2\left(\frac{\widetilde{v}}{c}\right)^4 \quad \langle T_3\rangle \sim \varepsilon^2 Mc^2,$$

where an average has been inserted since T_3 is fluctuating. It follows that

$$\frac{T_2}{T_1}\sim\left(\frac{\widetilde{v}}{c}\right)^2 \ll 1.$$

Thus, in the non-relativistic limit, T_2 is negligible in comparison to T_1. This is the case in a typical interferometry experiment where it can be $\widetilde{v}\approx 10^2$ m s^{-1} [11], so that $(\widetilde{v}/c)^2\sim 10^{-12}$. Next we have:

$$\frac{\langle T_3\rangle}{T_1}\sim\left(\frac{\varepsilon c}{\widetilde{v}}\right)^2.$$

The request that the conformal fluctuations have a small effect thus gives the condition

$$\frac{\langle T_3\rangle}{T_1}\ll 1 \quad\Leftrightarrow\quad \varepsilon^2\sim\left\langle A^2\right\rangle \ll \left(\frac{\widetilde{v}}{c}\right)^2. \tag{4}$$

That this condition is effectively satisfied can be checked a posteriori after the model is complete. It depends on the statistical properties of the conformal field and the particle ability to probe them. This will be related to a particle *resolution scale*. At the end of the discussion in section 8.3 we will show that (4) is satisfied if, e.g., the particle resolution scale is given by its Compton length.

Under these conditions the non-relativistic limit of equation (3) yields:

$$-\frac{\hbar^2}{2M}\nabla^2\psi+\left(A+\frac{A^2}{2}\right)Mc^2\psi+T_4=i\hbar\frac{\partial\psi}{\partial t}, \tag{5}$$

where

$$T_4 := -\frac{\hbar^2}{M}\partial^a\phi\partial_a\ln(1+A)\times\exp\left(iMc^2t/\hbar\right).$$

In order to assess the correction due to this term we split it into two contributions by writing separately the time and space derivatives. Using the fact that

$$\frac{\partial\phi}{\partial t}\approx -\frac{iMc^2}{\hbar}\psi\exp\left(-iMc^2t/\hbar\right)$$

it is easy to see that:

$$T_4 = -i\hbar\left(\dot{A}-A\dot{A}\right)\psi - i\hbar\widetilde{v}\left(A_{,x}-AA_{,x}\right)\psi, \tag{6}$$

where $\dot{A} := \partial A/\partial t$, $A_{,x} := \partial A/\partial x$ and where we assumed that the particle velocity in the laboratory is along the x axis. In Appendix B we show that *if A is (i) a stochastic isotropic perturbation and (ii) effectively fast varying over a typical length $\lambda_A = \kappa h/(Mc)$ related to the particle resolution scale*, then T_4 reduces to:

$$T_4 = \left(-A + A^2\right) \frac{Mc^2}{\kappa}\psi. \tag{7}$$

Here $\kappa \sim 1$ is dimensionless and its precise value is unimportant. The important point is that T_4 yields a *positive* extra nonlinear term in A that adds up to what we already have in (5). Finally we get the *effective Schrödinger equation*

$$-\frac{\hbar^2}{2M}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t},$$

where the *nonlinear fluctuating potential* V is defined by

$$V := \left(C_1 A + C_2 A^2\right) Mc^2. \tag{8}$$

The values of the constants C_1 and C_2 depend on κ and C_2 *is always strictly positive*. For $\kappa = 1$ it would be $C_1 = 0$ and $C_2 = 3/2$. For generality we will leave them unspecified in the following treatment and consider κ as a constant of order one.

3. Average quantum evolution

3.1. Dyson expansion for short evolution time

We now have a rather well defined problem: that of the dynamics of a non relativistic quantum particle under the influence of the nonlinear potential (8). The Schrödinger equation describing the dynamics of a free particle is suitable to describe the interference patterns that could result e.g. in an interferometry experiment employing cold molecular beams. When the particle in the beam propagates through an environment, we are dealing with an open quantum system. This in general suffers decoherence, resulting in a loss of visibility in the fringes pattern [9, 11]. This is a well defined macroscopic quantity. In the present semiclassical treatment the environment due to spacetime fluctuations is represented, down to the semiclassical scale ℓ, by a sea of random radiation encoded in A and resulting in the fluctuating potential V. An estimate of the overall dephasing can be obtained by considering the statistical averaged dynamics of a single quantum particle interacting with V. In practise we will need (i) to solve for the dynamics of a single particle of mass M and (ii) calculate the averaged wavefunction by averaging over the fluctuations. The outcome of (i) would be some sort of 'fluctuating' wavefunction carrying, beyond the information related to the innate quantum behavior of the system, that related to the fluctuations in the potential. The outcome of (ii) is to yield a general statistical result describing what would be obtained in an experiment where many identical particles propagate through the same fluctuating potential.

We thus consider the Hamiltonian operator $\hat{H}(t) = \hat{H}^0 + \hat{H}^1(t)$, where $\hat{H}^0$ is the kinetic part while

$$\hat{H}^1(t) = \int \mathrm{d}^3x V(\mathbf{x}, t)|\mathbf{x}\rangle\langle\mathbf{x}|,$$

is the perturbation due to the fluctuating potential energy. Here $|\mathbf{x}\rangle\langle\mathbf{x}|$ is the projection operator on the space spanned by the position operator eigenstate $|\mathbf{x}\rangle$. Indicating the state vector at time t with ψ_t, the related Schrödinger equation reads

$$\hat{H}(t)\psi_t = i\hbar\frac{\partial\psi_t}{\partial t}.$$

Using the density matrix formalism, the general solution can be expressed through a Dyson series as [36]

$$\rho_T = \rho_0 + \hat{K}_1(T)\rho_0 + \rho_0\hat{K}_1^\dagger(T) + \hat{K}_2(T)\rho_0 + \hat{K}_1(T)\rho_0\hat{K}_1^\dagger(T) + \rho_0\hat{K}_2^\dagger(T) + \ldots,$$

where ρ_0 is the initial density matrix and the propagators $\hat{K}_1(T)$ and $\hat{K}_2(T)$ are given by

$$\hat{K}_1(T) := -\frac{i}{\hbar}\int_0^T \hat{H}(t')\mathrm{d}t',$$

$$\hat{K}_2(T) := -\frac{1}{\hbar^2}\int_0^T \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \hat{H}(t')\hat{H}(t'').$$

In truncating the series to second order we assume that the system evolves for a time T such that $T \ll T^*$, where T^* is defined as the typical time scale required to have a significant change in the density matrix ρ.

The effect of the environment upon a large collection of identically prepared systems is found by taking the average over the fluctuating potential as explained above. Formally and up to second order we have

$$\langle\rho_T\rangle = \left\langle \rho_0 + \hat{K}_1(T)\rho_0 + \rho_0\hat{K}_1^\dagger(T) + \hat{K}_2(T)\rho_0 + \hat{K}_1(T)\rho_0\hat{K}_1^\dagger(T) + \rho_0\hat{K}_2^\dagger(T)\right\rangle.$$

The average density matrix $\langle\rho_T\rangle$ will describe the average evolution of the system including the effect of dephasing.

It is straightforward to show that, up to second order in the Dyson's expansion, the kinetic and potential parts of the hamiltonian give independent, additive contributions to the average evolution of the density matrix, i.e. $\langle\rho_T\rangle = [\rho_T]_0 + \langle[\rho_T]_1\rangle$, where

$$\begin{aligned}[\rho_T]_0 := \rho_0 &+ [\hat{K}_1(T)]_0\rho_0 + \rho_0[\hat{K}_1(T)]_0^\dagger \\ &+ [\hat{K}_2(T)]_0\rho_0 + [\hat{K}_1(T)]_0\rho_0[\hat{K}_1(T)]_0^\dagger + \rho_0[\hat{K}_2(T)]_0^\dagger,\end{aligned}$$

$$\begin{aligned}\langle[\rho_T]_1\rangle := \Big\langle \rho_0 &+ [\hat{K}_1(T)]_1\rho_0 + \rho_0[\hat{K}_1(T)]_1^\dagger \\ &+ [\hat{K}_2(T)]_1\rho_0 + [\hat{K}_1(T)]_1\rho_0[\hat{K}_1(T)]_1^\dagger + \rho_0[\hat{K}_2(T)]_1^\dagger\Big\rangle. \qquad (9)\end{aligned}$$

Here the kinetic propagators $[\hat{K}_1(T)]_0$ and $[\hat{K}_2(T)]_0$ depend solely on $\hat{H}^0$, while the potential propagators $[\hat{K}_1(T)]_1$ and $[\hat{K}_2(T)]_1$ depend only on $\hat{H}^1(t)$. In the next section we estimate the dephasing by calculating the term $\langle[\rho_T]_1\rangle$ alone.

4. The conformal field and its correlation properties

We now set the statistical properties of the conformal field A. This is assumed to represent a real, stochastic process having a zero mean. We further assume it to be isotropic. In Appendix A we review a series of important results concerning stochastic processes, in particular in relation to real stochastic signals satisfying the wave equation. The main quantity characterizing the process is the power spectral density $S(\omega)$. In the case of an isotropic bath of random radiation, field averages such as $\langle A^2\rangle$, $\langle|\nabla A|^2\rangle$ and $\langle(\partial_t A)^2\rangle$ can be found in terms of $S(\omega)$, e.g.

$$\left\langle A(\mathbf{x},t)^2\right\rangle = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, S(k),$$

where $kc = \omega$. In Appendix A we show how the conformal field can be resolved into components traveling along all possible space directions according to

$$A(\mathbf{x}, t) = \int \mathrm{d}\hat{\mathbf{k}}\, A_{\hat{\mathbf{k}}}(\hat{\mathbf{k}} \cdot \mathbf{x}/c - t),$$

where $\mathrm{d}\hat{\mathbf{k}}$ indicates the elementary solid angle. The capacity of the fluctuations to maintain correlation is encoded in the autocorrelation function $C(\tau)$. In the same appendix we prove a generalization of the usual Wiener-Khintchine theorem, valid for the case of a spacetime dependent process satisfying the wave equation, and linking the autocorrelation function to the Fourier transform of the power spectral density according to:

$$C(\tau) := \frac{1}{(2\pi c)^3} \int \mathrm{d}\omega\, \omega^2 S(\omega) \cos(\omega\tau). \tag{10}$$

This allows to prove that wave components traveling along independent space directions are uncorrelated, i.e.

$$\left\langle A_{\hat{\mathbf{k}}}(t)\, A_{\hat{\mathbf{k}}'}(t+\tau) \right\rangle = \delta(\mathbf{k}, \mathbf{k}')\, C(\tau). \tag{11}$$

The field mean squared amplitude is related to the correlation function according to $\left\langle A^2 \right\rangle = 4\pi C_0$, as derived in Appendix A

Isotropy implies that all directional components have the same amplitude A_0. This is found introducing the *normalized correlation function* $R(\tau)$ through

$$R(\tau) := \frac{C(\tau)}{C(0)},$$

so that $R(0) = 1$. Equation (11) can now be re-written as $\left\langle A_{\hat{\mathbf{k}}}(t)\, A_{\hat{\mathbf{k}}'}(t+\tau) \right\rangle = \delta(\mathbf{k}, \mathbf{k}')\, C_0\, R(\tau)$ so that, introducing the *normalized directional components*, $f_{\hat{\mathbf{k}}}(t) := A_{\hat{\mathbf{k}}}(t)/\sqrt{C_0}$ we have $\left\langle f_{\hat{\mathbf{k}}}(t)\, f_{\hat{\mathbf{k}}'}(t+\tau) \right\rangle = \delta(\mathbf{k}, \mathbf{k}')\, R(\tau)$. We now define the constant $A_0 := \sqrt{C_0}$ which is connected to the *squared amplitude per solid angle* according to $A_0^2 = C_0 = \left\langle A^2 \right\rangle / 4\pi$. The directional components are given by $A_{\hat{\mathbf{k}}}(t) = A_0\, f_{\hat{\mathbf{k}}}(t)$ and the general conformal field can finally be expressed as an elementary superposition of the kind

$$A(\mathbf{x}, t) = A_0 \int \mathrm{d}\hat{\mathbf{k}}\, f_{\hat{\mathbf{k}}}(\hat{\mathbf{k}} \cdot \mathbf{x}/c - t). \tag{12}$$

4.1. Summary of the correlation properties of the conformal fluctuations

The main statistical properties of the directional stochastic waves $f_{\hat{\mathbf{k}}}$ are summarized by

$$\left\langle f_{\hat{\mathbf{k}}}(t) \right\rangle = 0, \tag{13}$$

$$\left\langle f_{\hat{\mathbf{k}}}(t) f_{\hat{\mathbf{k}}'}(t') \right\rangle = \delta(\hat{\mathbf{k}}, \hat{\mathbf{k}}') R(t - t'), \tag{14}$$

i.e. each component has *zero mean* and fluctuations traveling along different space directions are perfectly *uncorrelated*. These two properties imply that odd products of directional components have also a zero mean, i.e.

$$\left\langle f_{\hat{\mathbf{k}}_1}(t_1) f_{\hat{\mathbf{k}}_2}(t_2) f_{\hat{\mathbf{k}}_3}(t_3) \right\rangle = 0. \tag{15}$$

In the following dephasing calculation we will need to evaluate means involving products of four directional components. To this purpose we need to introduce the *second order correlation function* $R''(t-t')$ according to

$$\left\langle [f_{\hat{\mathbf{k}}}(t)]^2[f_{\hat{\mathbf{k}}'}(t')]^2 \right\rangle = 1 + \delta(\hat{\mathbf{k}},\hat{\mathbf{k}}')[R''(t-t')-1]. \tag{16}$$

This definition is compatible with the fact that the mean is one when components traveling in different direction are involved, i.e. $\left\langle [f_{\hat{\mathbf{k}}}(t)]^2[f_{\hat{\mathbf{k}}'}(t')]^2 \right\rangle = 1$ if $\hat{\mathbf{k}} \neq \hat{\mathbf{k}}'$.

5. Dephasing calculation outline

To calculate the dephasing suffered by the probing particle we must evaluate the average of all the individual terms in equation (9). The relevant propagators are

$$[\hat{K}_1(T)]_1 := -\frac{i}{\hbar}\int_0^T \mathrm{d}t'\,\hat{H}^1(t'),$$

$$[\hat{K}_2(T)]_1 := -\frac{1}{\hbar^2}\int_0^T \mathrm{d}t \int_0^t \mathrm{d}t'\,\hat{H}^1(t)\hat{H}^1(t').$$

The interaction Hamiltonian is given by

$$\hat{H}^1(t) = \int \mathrm{d}^3x V(\mathbf{x},t)|\mathbf{x}\rangle\langle\mathbf{x}|,$$

where the potential energy is

$$V(\mathbf{x},t) = C_1Mc^2A_0\int \mathrm{d}\hat{\mathbf{k}}\, f_{\hat{\mathbf{k}}}(t-\mathbf{x}\cdot\hat{\mathbf{k}}/c) + C_2Mc^2A_0^2\left[\int \mathrm{d}\hat{\mathbf{k}}\, f_{\hat{\mathbf{k}}}(t-\mathbf{x}\cdot\hat{\mathbf{k}}/c)\right]^2.$$

5.1. First order terms of the Dyson expansion

We evaluate the two first order terms in the Dyson expansion. For a more compact notation, we do not show the argument of the directional components $f_{\hat{\mathbf{k}}}$. The contribution of the linear part of the potential C_1Mc^2A vanishes trivially since $\langle A\rangle = 0$. The quadratic part gives:

$$\left\langle \hat{K}_1(T)\rho_0 \right\rangle = -\frac{iC_2Mc^2A_0^2}{\hbar}\int_0^T \mathrm{d}t \int \mathrm{d}^3x\,|\mathbf{x}\rangle\langle\mathbf{x}| \left\langle \left[\int \mathrm{d}\hat{\mathbf{k}}\, f_{\hat{\mathbf{k}}}\right]^2 \right\rangle \rho_0.$$

Using $A_{\hat{\mathbf{k}}}(t) = \sqrt{C_0}f_{\hat{\mathbf{k}}}(t)$ and $\left\langle A^2\right\rangle = 4\pi C_0 \equiv 4\pi A_0^2$ it is seen that the average yields 4π. Since $\int \mathrm{d}^3x\,|\mathbf{x}\rangle\langle\mathbf{x}| = \hat{\mathrm{I}}$ and integrating over T we find

$$\left\langle \hat{K}_1(T)\rho_0 \right\rangle = -\frac{4\pi C_2 iMc^2A_0^2T}{\hbar}\rho_0. \tag{17}$$

The calculation of the other first order term proceeds in the same way. Since $\hat{K}_1^\dagger(T) = -\hat{K}_1$, it yields the same result as in (17) but with the opposite sign (more in general, all the odd terms in the Dyson expansion have an i factor and also yield a vanishing contribution). We thus see that *at first order in the Dyson expansion there is no net dephasing and* $\langle \hat{K}_1(T)\rho_0 + \rho_0\hat{K}_1^\dagger(T)\rangle = 0$.

5.2. Second order terms of the Dyson expansion

The second order calculation is more complicated. A fundamental point is that the linear part of the potential does again give a vanishing contribution. Dephasing will be shown to come as a purely nonlinear effect due to the nonlinear potential term $\sim A^2$.

5.2.1. (Non)-contribution of the linear part of the potential To have an idea of how things work we consider e.g. the average of the term $\hat{K}_2\rho_0$. This has the following structure:

$$\left\langle \hat{K}_2\rho_0 \right\rangle \sim \int \mathrm{d}t \int \mathrm{d}t' \int \mathrm{d}^3y |\mathbf{y}\rangle\langle\mathbf{y}| \int \mathrm{d}^3y' |\mathbf{y}'\rangle\langle\mathbf{y}'| \left\langle V(\mathbf{y},t)V(\mathbf{y}',t')\right\rangle \rho_0.$$

The interesting part is the average $\langle V(\mathbf{y},t)V(\mathbf{y}',t')\rangle$. This is:

$$\langle V(\mathbf{y},t)V(\mathbf{y}',t')\rangle \sim \langle A(\mathbf{y},t)A(\mathbf{y}',t')\rangle + \left\langle A(\mathbf{y},t)A^2(\mathbf{y}',t')\right\rangle + \left\langle A^2(\mathbf{y},t)A^2(\mathbf{y}',t')\right\rangle.$$

The first two term are due to the linear part of the potential. The second of them vanishes in virtue of property (15). This is seen using the directional decomposition (12) and writing:

$$\left\langle A(\mathbf{y},t)A^2(\mathbf{y}',t')\right\rangle = A_0^3 \int \mathrm{d}\hat{\mathbf{k}}_1 \int \mathrm{d}\hat{\mathbf{k}}_2 \int \mathrm{d}\hat{\mathbf{k}}_3 \left\langle f_{\hat{\mathbf{k}}_1} f_{\hat{\mathbf{k}}_2} f_{\hat{\mathbf{k}}_3} \right\rangle = 0.$$

The first term results in the contribution:

$$\langle A(\mathbf{y},t)A(\mathbf{y}',t')\rangle \Rightarrow$$
$$\int_0^T \mathrm{d}t \int_0^t \mathrm{d}t' \int \mathrm{d}^3y |\mathbf{y}\rangle\langle\mathbf{y}| \int \mathrm{d}^3y' |\mathbf{y}'\rangle\langle\mathbf{y}'| \int \mathrm{d}\hat{\mathbf{k}} \int \mathrm{d}\hat{\mathbf{k}}' \left\langle f_{\hat{\mathbf{k}}}(t-\mathbf{y}\cdot\hat{\mathbf{k}}) f_{\hat{\mathbf{k}}'}(t'-\mathbf{y}'\cdot\hat{\mathbf{k}}') \right\rangle \rho_0.$$

For convenience of notation we set $c=1$ in the arguments of the directional functions $f_{\hat{\mathbf{k}}}$. Using equation (14) the average yields the 2-point correlation function according to $\delta(\hat{\mathbf{k}},\hat{\mathbf{k}}')R(t-t'+\mathbf{y}'\cdot\hat{\mathbf{k}}'-\mathbf{y}\cdot\hat{\mathbf{k}})$. Integrating with respect to $\hat{\mathbf{k}}'$ yields:

$$\langle A(\mathbf{y},t)A(\mathbf{y}',t')\rangle \Rightarrow$$
$$\int_0^T \mathrm{d}t \int_0^t \mathrm{d}t' \int \mathrm{d}^3y |\mathbf{y}\rangle\langle\mathbf{y}| \int \mathrm{d}^3y' |\mathbf{y}'\rangle\langle\mathbf{y}'| \int \mathrm{d}\hat{\mathbf{k}} R[t-t'+\hat{\mathbf{k}}\cdot(\mathbf{y}'-\mathbf{y})]\rho_0.$$

The corresponding matrix element is found by inserting $\langle\mathbf{x}|$ and $|\mathbf{x}'\rangle$ respectively on the left and on the right. Using $\langle\mathbf{x}|\mathbf{y}\rangle = \delta(\mathbf{x}-\mathbf{y})$ and exploiting the properties of the delta function we find

$$\langle A(\mathbf{y},t)A(\mathbf{y}',t')\rangle \Rightarrow K \times \int_0^T \mathrm{d}t \int_0^t \mathrm{d}t' \int \mathrm{d}\hat{\mathbf{k}} R(t-t'), \tag{18}$$

where K is a constant given by

$$K = -\frac{C_1^2 A_0^2 M^2 c^4 \rho_{\mathbf{x}\mathbf{x}'}(0)}{\hbar^2},$$

and where $\rho_{\mathbf{x}\mathbf{x}'}(0) := \langle\mathbf{x}|\rho_0|\mathbf{x}'\rangle$. The similar terms coming from $\left\langle \rho_0\hat{K}_2^\dagger \right\rangle$ will contribute in the same way as in (18), thus yielding an extra factor 2. Finally, through a similar calculation it is found that the terms $\sim A(\mathbf{y},t)A(\mathbf{y}',t')$ coming from $\left\langle \hat{K}_1\rho_0\hat{K}_1^\dagger \right\rangle$ contribute according to:

$$\langle A(\mathbf{y},t)A(\mathbf{y}',t')\rangle \Rightarrow -K \times \int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' \int \mathrm{d}\hat{\mathbf{k}} R[t-t'+\hat{\mathbf{k}}\cdot(\mathbf{x}'-\mathbf{x})].$$

Bringing all together, the overall contribution deriving from the linear part $C_1 Mc^2 A$ of the effective potential is found to be proportional to the expression:

$$I := \int_0^T \mathrm{d}t \left\{ 2\int_0^t \mathrm{d}t' R(t-t') - \int_0^T \mathrm{d}t' R[t-t'+\hat{\mathbf{k}}\cdot(\mathbf{x}'-\mathbf{x})] \right\}. \quad (19)$$

In Appendix C we prove that this vanishes provided $R(\tau)$ is an even function and provided that the drift time T is much larger than the time needed by the fluctuations to propagate through the distance $|\Delta\mathbf{x}|$, i.e. if $T \gg \hat{\mathbf{k}}\cdot\Delta\mathbf{x}$, where $c = 1$. This condition is certainly satisfied in a typical interferometry experiment where the drift time T can be of the order of ~ 1 ms and cT is indeed much larger than the typical space separations $|\Delta\mathbf{x}|$ relevant to quantify the loss of contrast in the measured interference pattern.

Thus we have here the important result that the linear part of the potential doesn't induce in general any dephasing up to second order in Dyson expansion. In fact we show in the next section that dephasing results *purely* as an effect of the nonlinear potential term $C_2 Mc^2 A^2$.

5.2.2. Contribution of the nonlinear part of the potential This calculation requires estimating averages of the kind $\langle A^2(\mathbf{y},t)A^2(\mathbf{y}',t')\rangle$, which will bring in the second order correlation function R'' defined in (16). This is straightforward but algebraically lengthy. Proceeding in a similar way as done above, exploiting the statistical properties (13)-(16) and the already mentioned result $I = 0$ in relation to (19), then the general result for the density matrix and valid up to second order in the Dyson expansion can be proved to be:

$$\begin{aligned}\rho_{\mathbf{x}\mathbf{x}'}(T) = {} & \rho_{\mathbf{x}\mathbf{x}'}(0) - \frac{32C_2\pi^2M^2c^4A_0^4\rho_{\mathbf{x}\mathbf{x}'}(0)}{\hbar^2} \times \Bigg[\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' R^2(t-t') \\ & -\frac{1}{16\pi^2}\int \mathrm{d}\hat{\mathbf{k}} \int \mathrm{d}\hat{\mathbf{K}} \int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' R(t-t'-\hat{\mathbf{k}}\cdot\Delta\mathbf{x}/c)\times R(t-t'-\hat{\mathbf{K}}\cdot\Delta\mathbf{x}/c)\Bigg]. \quad (20)\end{aligned}$$

Remarkably, *the second order correlation function doesn't play any role: the first order correlation function $R(\tau)$ and thus the power spectral density $S(\omega)$ completely determine the system evolution up to second order.* Equation (20) implies that the diagonal elements of the density matrix are left unchanged by time evolution. This is seen by setting $\Delta\mathbf{x} = 0$ which yields immediately $\rho_{\mathbf{x}\mathbf{x}}(T) = \rho_{\mathbf{x}\mathbf{x}}(0)$ for every T.

6. General density matrix evolution for large drift times

To verify that we have dephasing with an exponential decay of the off diagonal elements we need further simplify the result (20) by analyzing its behavior for appropriately large evolution times. To this end we start from the following identity

$$\begin{aligned}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' g(t-t') &= \frac{1}{2\pi}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{g}(\omega) e^{i\omega(t-t')} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{g}(\omega)\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2,\end{aligned}$$

where $\tilde{g}(\omega)$ denotes the Fourier transform of the function $g(t)$. Denoting $[0, \Delta\omega]$ as a frequency interval where $\tilde{g}(\omega)$ is slow varying, it is straightforward to show that the above identity reduces to

$$\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' g(t-t') \approx \tilde{g}(0)T, \tag{21}$$

for $T \gtrsim (\Delta\omega)^{-1}$. Note that for this to happen $g(t)$ doesn't even need being an even function. This condition translates what we mean by appropriately long evolution time. In section 8.2 we will show that it is equivalent to $T \gtrsim \tau_*$, where τ_* is the fluctuations correlation time. This is defined below.

Equation (21) can now be used to evaluate the time integrals appearing in (20). This is done by identifying in one case $g(t) := R^2(t)$ and in the other $g_{\tau\tau'}(t) := R(t+\tau)R(t+\tau')$, where τ and τ' stand respectively for $-\hat{\mathbf{k}} \cdot \Delta\mathbf{x}$ and $-\hat{\mathbf{K}} \cdot \Delta\mathbf{x}$, and where the normalized correlation function can be expressed, using the generalized W-K theorem (10), as

$$R(\tau) \equiv \frac{C(\tau)}{C_0} = \frac{1}{C_0(2\pi c)^3} \int_0^{\omega_c} \mathrm{d}\omega \, \omega^2 S(\omega) \cos(\omega\tau).$$

Notice that the integration frequency has a cutoff at $\omega_c = \omega_P/\lambda$, where the Planck frequency is $\omega_P := 2\pi/T_{\mathrm{P}} = 1.166 \times 10^{44}\ \mathrm{s}^{-1}$. This is consistent with the fact that below the scale $\ell = \lambda L_P$ the approximation of randomly fluctuating fields breaks down. In alternative this may simply correspond to the fact that the probing particle is insensitive to the short wavelengths as a result of its own finite resolution scale L_{R}.

Application of (21) to $g(t) := R^2(t)$ yields the result

$$\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' R^2(t-t') = \tau_* T, \tag{22}$$

where the *correlation time* is defined as

$$\tau_* := \mathfrak{F}\left[R^2(t)\right](0) = \pi \frac{\int_0^{\omega_c} \mathrm{d}\omega \, \omega^4 \, S^2(\omega)}{\left[\int_0^{\omega_c} \mathrm{d}\omega \, \omega^2 \, S(\omega)\right]^2}, \tag{23}$$

$\mathfrak{F}$ denoting Fourier transform. On the other hand, application of (21) to $g_{\tau\tau'}(t) := R(t+\tau)R(t+\tau')$ gives the result

$$\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' R(t+\tau)R(t+\tau') = \tau_* \Gamma[\omega_c(\tau-\tau')] \, T, \tag{24}$$

where the *characteristic function* Γ has been defined as

$$\Gamma(\omega_c t) := \frac{\int_0^{\omega_c} \mathrm{d}\omega \, \omega^4 S^2(\omega) \cos(\omega t)}{\int_0^{\omega_c} \mathrm{d}\omega \, \omega^4 S^2(\omega)}. \tag{25}$$

This is dimensionless and satisfies in general the following properties:

- $\Gamma(\omega_c t) = \Gamma(-\omega_c t)$,
- $\Gamma(0) = 1$,
- $\Gamma(\omega_c t) < 1, \quad \text{for} \quad t \neq 0$,
- $\Gamma(\omega_c t) \to 0, \quad \text{for} \quad t \to \infty$.

Notice that both the correlation time τ_* and the characteristic function Γ solely depend on the fluctuations power spectral density.

The results (22) and (24) can now be used in equation (20) to yield the neat result

$$\rho_{\mathbf{x}\mathbf{x}'}(T) = \rho_{\mathbf{x}\mathbf{x}'}(0)\left[1 - \frac{32C_2\pi^2 M^2 c^4 A_0^4 \tau_* T}{\hbar^2} \times F(\Delta\mathbf{x})\right], \tag{26}$$

where

$$F(\Delta\mathbf{x}) := 1 - \frac{1}{16\pi^2}\int \mathrm{d}\hat{\mathbf{k}}\int \mathrm{d}\hat{\mathbf{K}}\,\Gamma[\omega_c(\hat{\mathbf{K}} - \hat{\mathbf{k}})\cdot\Delta\mathbf{x}/c]. \tag{27}$$

This equation is important and represents one of the main result of this paper. It implies that *dephasing due to conformal fluctuations does indeed occur in general and independently of the precise power spectrum characterizing the fluctuations.* Without the need to evaluate the angular integrals, this follows from the properties of the characteristic function Γ. The fact that $\Gamma[\omega_c t] < 1$ implies $0 \leq F(\Delta\mathbf{x}) \leq 1$ with (i) $F(\Delta\mathbf{x} = 0) = 0$ and (ii) $F(\Delta\mathbf{x} \to \infty) = 1$ as special limiting cases. As a consequence the diagonal elements are unaffected while the off-diagonal elements decay exponentially according to

$$\dot{\rho}_{\mathbf{x}\mathbf{x}'}(0) := \frac{\rho_{\mathbf{x}\mathbf{x}'}(T) - \rho_{\mathbf{x}\mathbf{x}'}(0)}{T} = -\left[\frac{32C_2\pi^2 M^2 c^4 A_0^4 \tau_*}{\hbar^2} \times F(\Delta\mathbf{x})\right]\rho_{\mathbf{x}\mathbf{x}'}(0),$$

providing of course that T is small enough so that the change in the density matrix is small. Finally, if $\delta\rho := \rho_{\mathbf{x}\mathbf{x}'}(T) - \rho_{\mathbf{x}\mathbf{x}'}(0)$, we can define the *dephasing rate* as $|\delta\rho/\rho_0|$. Thanks to the property $F(\Delta\mathbf{x} \to \infty) = 1$, this converges for large spacial separations to the constant maximum value

$$\left|\frac{\delta\rho}{\rho_0}\right| = \frac{32C_2\pi^2 M^2 c^4 A_0^4 \tau_* T}{\hbar^2}. \tag{28}$$

This result based on the present 3-dimensional analysis of the conformal fluctuations can be compared to the analogue 1-dimensional result that PP found in [1]. Using a gaussian correlation function from the outset they found

$$\left|\frac{\delta\rho}{\rho_0}\right|_{1D} = \frac{\sqrt{\pi}M^2 c^4 A_0^4 \tau_g T}{\sqrt{2}\hbar^2},$$

where τ_g stands for some characteristic correlation time of the fluctuations. Identifying approximately $\tau_* \simeq \tau_g$, we have $(32C_2\pi^2)/(\sqrt{\pi/2}) \simeq 250$, assuming $C_2 \sim 1$. *Thus the present 3-dimensional analysis is seen to predict a dephasing rate 2 orders of magnitude larger than in the idealized 1-dimensional case.*

6.1. A remark on the validity of the Dyson expansion

We have found that the change in the density matrix is given by:

$$T_2[A_0^4] \sim \left(\frac{Mc^2}{\hbar}\right)^2 \tau_* T \times A_0^4,$$

In order for the expansion scheme to be effective, the propagation time T must be short enough to guarantee that $T_2[A_0^4]$ is small. How short depends of course on the statistical properties of the fluctuations, encoded in τ_*, and on the probing particle mass M. A fullerene C_{70} molecule with ($M_{C_{70}} \approx 10^{-24}$ kg) gives $Mc^2/\hbar \approx 10^{27}\mathrm{s}^{-1}$. Therefore the approach is consistent only if correlation time τ_*, the flight time T and field squared amplitude A_0^2 are appropriately small. We will come back on this issue

in section 8.2, where it is shown that, in the case of vacuum fluctuations (introduced in the next session), it is $\tau_* \sim \lambda T_{\rm P}$ and $A_0^2 \sim 1/\lambda^2$. For a flight time $T \approx 1$ ms, typical of interferometry experiments, this results in $T_2[A_0^4] \approx 10^7/\lambda^{-3}$. For any reasonable value of $\lambda \gtrsim 10^3$ the density matrix change is indeed small and the Dyson expansion scheme well posed up to second order. In Appendix D we estimate the fourth order term in the expansion, which will also yield a term proportional to A_0^4. It will be shown that its contribution in fact vanishes under quite general circumstances. This puts the result (26) on an even stronger basis.

7. Explicit dephasing rate in the case of vacuum fluctuations

The result (26) is quite general. The only ingredients entering the analysis so far have been: (i) a spacetime metric $g_{ab} = (1+A)^2\eta_{ab}$ with $A = O(\varepsilon \ll 1)$, (ii) a randomly fluctuating conformal field A satisfying the wave equation $\partial^c\partial_c A = 0$, and (iii) the isotropic fluctuations characterized by an arbitrary power spectral density $S(\omega)$. The dephasing then occurs as a result of the nonlinearity in the effective potential $V = Mc^2[C_1 A + C_2 A^2]$.

A particularly interesting case, potentially related to the possibility of detecting experimental signs of quantum gravity, is that in which the fluctuations in A are the manifestation, at the appropriate semiclassical scale $\ell = \lambda L_P$, of underlying *vacuum quantum fluctuations* close to the Planck scale. Strictly speaking the presence of the probing particle perturbs the genuine quantum vacuum state. For this reason it would be appropriate to talk of *effective vacuum*, i.e. up to the presence of the test particle. By its nature, the present semiclassical analysis *cannot* take in account the backreaction of the system on the environment. Therefore we simply assume that the modifications on the vacuum state can be neglected as long as the probing particle mass is not too large and the evolution time short. We thus model the effective vacuum properties of the conformal field A at the semiclassical scale on the basis of the properties that real vacuum is expected to possess at the same scale. It is a fact that vacuum looks the same to all inertial observers far from gravitational fields. In particular, its energy density content should be Lorentz invariant. This can obtained through an appropriate choice of the power spectrum $S(\omega)$.

7.1. Isotropic power spectrum for 'vacuum' conformal fluctuations

According to the above discussion we expect the average properties of A above the scale ℓ to be Lorentz invariant. In particular, the interesting quantities derived in section Appendix A.5

$$\left\langle A^2 \right\rangle = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, S(k),$$

$$\left\langle |\nabla A|^2 \right\rangle = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, k^2 S(k),$$

$$\left\langle (\partial_t A)^2 \right\rangle = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, \omega^2(k) S(k),$$

should be invariant. As discussed in Appendix A, for a stationary, isotropic signal, the averages $\langle\cdot\rangle$ can in fact be carried out through suitable spacetime integrations over an appropriate averaging scale $L \gg \ell$. In alternative they can be expressed as in the

above integrals depending on the power spectrum and adopting a high energy cutoff set by $k_\lambda := 2\pi/(\lambda L_P)$.

The problem of the Lorentz invariance of the above quantities has been discussed in details by Boyer [14] within his random electrodynamics framework. He showed that the choice

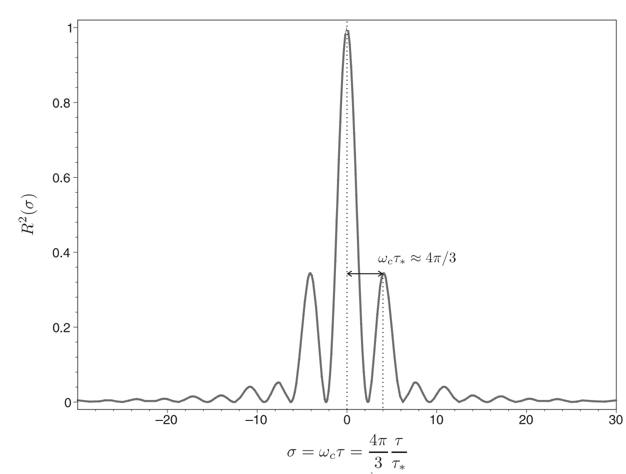
$$S(k) := \frac{\hbar G}{2c^2}\frac{1}{\omega(k)}, \tag{29}$$

guarantees a Lorentz invariant measure $\mathrm{d}^3k/\omega(k)$ (see also [37]) and implies an energy spectrum $\varrho(\omega) \propto \omega^3$, also shown to be the only possible choice for a Lorentz invariant energy spectrum of a massless field. The combination of the constants $\hbar$, G and c gives the correct dimensions for a power spectrum (i.e. L^3), while the factor $1/2$ guarantees that the resulting energy density is equivalent to that resulting from the superposition of zero-point contributions $\hbar\omega/2$. A final important point, which should not be overlooked, is that Lorentz invariance is preserved provided the cutoff k_λ is given by the *same number* for all inertial observers, as also discussed in details by Boyer. In other words this means that the critical length that sets the border line between the random field approach and the full quantum gravity regime is supposed to be the same for *any* inertial observer. It represents some kind of structural property of spacetime and *not* an observer dependent property. Accordingly it must not be transformed under Lorentz transformations. It is important to note that this requirement will also be satisfied when we employ an effective cutoff set by the particle Compton length.

Using (29) the normalized correlation function can be found explicitly from the generalized Wiener-Khintchine to be

$$R(\tau) := \frac{C(\tau)}{C_0} = 2\left[\frac{\sin\omega_c\tau}{\omega_c\tau} + \frac{\cos\omega_c\tau - 1}{\omega_c^2\tau^2}\right]. \tag{30}$$

Figure 1. *Plot of $R^2(t-t')$. The adimensional variable σ is basically $t-t'$ in units of the correlation time τ_*. It is seen that $\tau \approx \tau_*$ corresponds to the first of the secondary peaks.*



The peak of the autocorrelation function is linked to the fluctuations squared amplitude and gives explicitly:

$$C(0) \equiv A_0^2 = \frac{1}{8\pi\lambda^2}, \tag{31}$$

implying $\langle A^2 \rangle = 1/(2\lambda)^2$. The correlation time and characteristic function follow from equations (23) and (25) as:

$$\tau_* = \frac{2}{3}\lambda T_{\mathrm{P}}, \tag{32}$$

and

$$\Gamma(\sigma) = \frac{3\sin(\sigma)}{\sigma} + \frac{6\cos(\sigma)}{\sigma^2} - \frac{6\sin(\sigma)}{\sigma^3}, \tag{33}$$

where $\sigma = \omega_c t$ is a dimensionless variable. The plot of the squared normalized correlation function $R^2(t-t')$ is shown in figure 1: $t - t' = \tau_*$ corresponds to the first secondary peak in the curve, where the correlation in the fluctuations is reduced of $\sim 70\%$. This fully motivates the choice of τ_* to represent the correlation time.

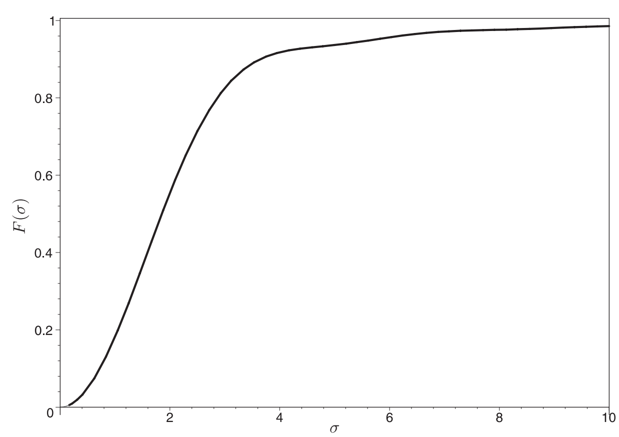
The explicit form of the characteristic function can be used in (27) to evaluate the remaining angular integrals and find the detailed expression for the density matrix evolution valid for all $(\mathbf{x}, \mathbf{x}')$. Isotropy implies that the result must depend on $|\mathbf{x} - \mathbf{x}'|$ only. The integration is straightforward and yields the result:

$$F(\sigma) := 1 - \frac{3}{2\sigma^2}\left(1 - \frac{\sin\sigma\cos\sigma}{\sigma}\right). \tag{34}$$

Substituting the results (31), (32), (33) and (34) into (26) yields the explicit result for the dephasing rate, valid for vacuum fluctuations described by $S \sim 1/\omega$:

$$\left|\frac{\delta\rho_{\mathbf{x}\mathbf{x}'}}{\rho_0}\right| = \frac{1}{3\lambda^3}\left(\frac{M}{M_P}\right)^2\left(\frac{T}{T_P}\right) \times F\left(\frac{2\pi\,|\mathbf{x} - \mathbf{x}'|}{\ell}\right), \tag{35}$$

Figure 2. *Plot of the function $F(\sigma)$ in the range $\sigma = 0..10$, where $\sigma = 2\pi\,|\mathbf{x} - \mathbf{x}'|\,/(\lambda L_P)$. The curve tends very rapidly to the limiting value of 1 and for spacial separations $|\mathbf{x} - \mathbf{x}'|$ which are slightly larger than $\ell = \lambda L_P$ the dephasing rate converges rapidly to its maximum value.*



where we considered $C_2 \sim 1$ and where

$$M_{\mathrm{P}} := \frac{\hbar}{c^2 T_{\mathrm{P}}} = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\mathrm{kg} = 1.310 \times 10^{19}\mathrm{amu}$$

is the Planck mass. The function F is plotted in figure 2. It enjoys the properties $F(0) = 0$ and $F(\sigma) \to 1$ for $\sigma \gg 1$, so that for $|\mathbf{x} - \mathbf{x}'| \gtrsim 10\ell$ the decoherence rate converges rapidly to its maximum value.

8. Discussion

8.1. Probing particle resolution scale and effective dephasing rate

Equation (35) is an important result. It gives the dephasing rate in the density matrix of a quantum particle propagating in space under the only action of a randomly fluctuating potential due to spacetime vacuum conformal fluctuations. The fact that it predicts an exponential decay of the off diagonal elements of the system density matrix (which is the distinctive feature of genuine quantum decoherence) is interesting as a further confirmation that certain effects involving quantum fluctuations can be mimicked by means of a semi-classical treatment in the spirit of Boyer [14, 15].

A significant feature of our dephasing formula is the quadratic dependence on the probing particle mass M, which comes as a consequence of the underlying non linearity. The coefficient $1/(3\lambda^3)$ sets the overall strength of the effect. It is proportional to A_0^4 and to the fluctuations correlation time τ_*: the more intense the fluctuations, the larger the dephasing and the longer the various directional component stay correlated, the higher their ability to induce dephasing. We have found $\tau_* \approx \lambda T_{\mathrm{P}}$, in such a way that the correlation time directly depends on the spacetime intrinsic cutoff parameter λ. According to this picture *all* the wavelength down to the cutoff $\ell = \lambda L_P$ should be able to affect the probing particle. However an atom or molecule is likely to possess its own resolution scale L_{R}. Thus, whenever $\tau_* c < L_{\mathrm{R}}$, the ability of the fluctuations to affect the particle would be reduced, as they would effectively average out. To characterize this feature of the problem we write, in analogy to $\ell = \lambda L_P$,

$$L_{\mathrm{R}} = \lambda_{\mathrm{R}} L_P,$$

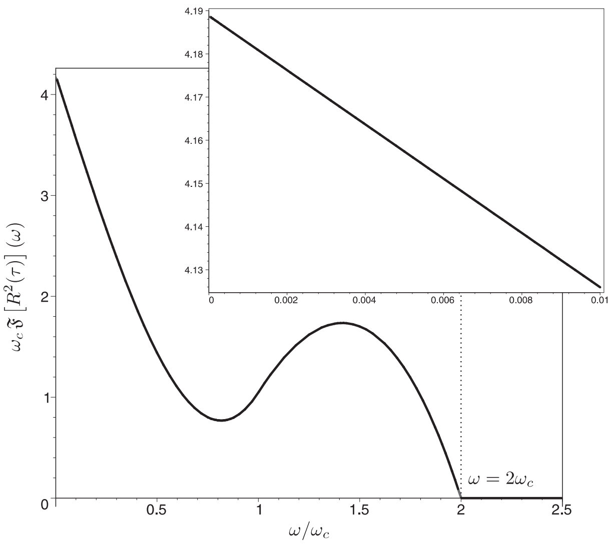
and use λ_{R} as a new, *particle dependent*, cutoff parameter. In general it is $\lambda_{\mathrm{R}} > \lambda$. The new effective correlation time is now given by $\tau_* \approx \lambda_{\mathrm{R}} T_{\mathrm{P}}$. The distance traveled by the fluctuations during a correlation time is $L_* = c\tau_* \equiv 2L_{\mathrm{R}}/3$. Thus the effective *correlation distance* L_* basically corresponds to the particle resolution scale: short wavelengths that do not keep their correlation up up the scale L_{R} average out and cannot affect the probing particle. The new, effective dephasing rate results by substituting λ with λ_{R} in (35):

$$\left|\frac{\delta\rho_{\mathbf{x}\mathbf{x}'}}{\rho_0}\right| = \frac{1}{3}\left(\frac{L_P}{L_{\mathrm{R}}}\right)^3 \left(\frac{M}{M_P}\right)^2 \left(\frac{T}{T_P}\right) \times F\left(\frac{2\pi\,|\mathbf{x}-\mathbf{x}'|}{L_{\mathrm{R}}}\right).$$

8.2. Validity of the long drift time regime

We recall that this results holds for 'long' drift times T, i.e. when $T \gtrsim (\Delta\omega)^{-1}$, where $\Delta\omega$ is an appropriate frequency range over which the Fourier transforms of $R^2(t)$ and $R(t+\tau)R(t+\tau')$ vary little. We are now in the position to make this precise and

Figure 3. *Fourier transform of $R^2(\sigma)$ as a function of the frequency ω in units of the cutoff frequency ω_c.*



define clearly the limits of applicability of the theory. To this end we consider the Fourier transform of $R^2(\omega_c\tau)$:

$$\mathfrak{F}[R^2(\omega_c\tau)](\omega) = \frac{1}{\omega_c}\mathfrak{F}[R^2(\sigma)](\omega/\omega_c),$$

with $R(\sigma)$ given in (30). Its plot is displayed in figure 3. The spectrum falls to 0 for $\omega \geq 2\omega_c$. The value of the peak at $\omega = 0$ is precisely $4\pi/3$, verifying that $\tau_* \equiv \mathfrak{F}[R^2(\omega_c\tau)](0) = 4\pi/(3\omega_c)$. The smaller box shows a zoom of the plot in the region $\sigma \in [0, 1/100]$: the curve is slow varying in this range since $\mathfrak{F}[R^2(\omega_c\tau)](0) = 4\pi/3 \approx 4.19$ and $\mathfrak{F}[R^2(\omega_c\tau)](1/100) \approx 4.13$. Similarly it is possible to check that the Fourier transform of $R(\sigma+\eta)R(\sigma+\eta')$, where the adimensional parameters η and η' depend on space direction and locations as $\eta := -\omega_c\hat{\mathbf{k}}\cdot\Delta\mathbf{x}/c$ and $\eta' := -\omega_c\hat{\mathbf{K}}\cdot\Delta\mathbf{x}/c$, enjoys a similar property: for *every* choice of η and η', the resulting Fourier transform is slow varying in the range $\sigma \in [0, 1/100]$. Following this discussion we chose $\Delta\omega \approx [0, \omega_c/100]$. We can now quantify the concept of 'long drift time' by $T \gtrsim 100/\omega_c$. From $\omega_c = 2\pi/(\lambda_{\rm R}T_{\rm P}) = 4\pi/(3\tau_*)$ this yields the condition

$$T \gtrsim 25\,\tau_*.$$

8.3. Some numerical estimates and outlook

In summary we have studied the dephasing on a non-relativistic quantum particle induced by a conformally modulated spacetime $g_{ab} = (1+A)^2\eta_{ab}$, where A is a random scalar field satisfying the wave equation. The important case of vacuum fluctuations can be characterized by a suitable power spectrum $S \sim 1/\omega$. If $L_{\rm R} = \lambda_{\rm R}L_P$ is the probing particle resolution scale, the dephasing rate for $|\mathbf{x}-\mathbf{x}'| \gg L_{\rm R}$ converges rapidly to:

$$\left|\frac{\delta\rho}{\rho_0}\right| = \frac{1}{3}\left(\frac{L_P}{L_{\rm R}}\right)^3\left(\frac{M}{M_P}\right)^2\left(\frac{T}{T_P}\right). \qquad (36)$$

The effective correlation time of the fluctuations is given by $\tau_* \approx \lambda_{\mathrm{R}} T_{\mathrm{P}}$. The above result holds for 'long' drift times satisfying

$$T \gtrsim (10 - 10^2)\lambda_{\mathrm{R}} T_{\mathrm{P}}.$$

To conclude we want to give some numerical estimates of the dephasing that could be expected in a typical matter wave interferometry experiment, e.g. like those described in [9], where fullerene molecules have been employed with drift times of the order of $T =: T_{\mathrm{ex}} \approx 10^{-3}$ s. Consider e.g. a C_{70} molecule with $M = M_{C_{70}} \approx 1.24 \times 10^{-24}$ kg. In comparison to the Planck units we have:

$$T_{\mathrm{ex}} \approx 10^{40} T_{\mathrm{P}}, \quad M_{C_{70}} \approx 10^{-17} M_{\mathrm{P}}.$$

Thus, it is clear that the most critical factor controlling the strength of the effect is set by the probing particle mass, together with the effective resolution cutoff scale. Using these data in (36) we can estimate:

$$\left|\frac{\delta\rho}{\rho_0}\right| \approx \frac{10^6}{\lambda_{\mathrm{R}}^3} \quad \Leftrightarrow \quad \lambda_{\mathrm{R}} \approx \left(\frac{10^6}{|\delta\rho/\rho_0|}\right)^{\frac{1}{3}}.$$

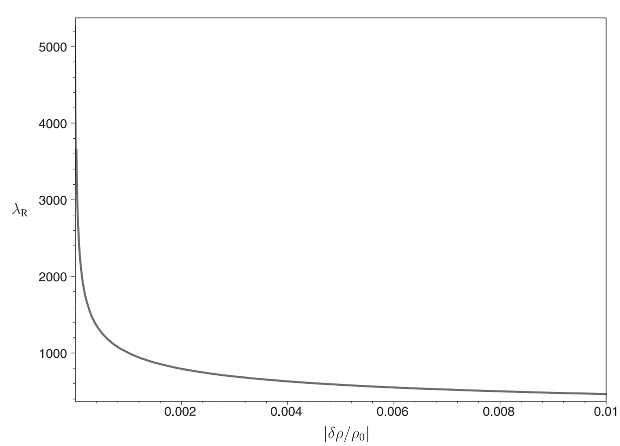
This could be used to estimate λ_{R} *if we were able to identify within an experiment a residual amount of dephasing that cannot be explained by other standard mechanisms* (e.g. environmental decoherence, internal degrees of freedom). Figure 4 plots λ_{R} against $|\delta\rho/\rho_0|$: a dephasing rate due to conformal fluctuations in the range $1\% - 0.1\%$ would imply a resolution parameter in the range $\lambda_{\mathrm{R}} \approx 10^3 - 10^4$. This would represent a lower bound on λ_{R}, as interferometry experiments will get more and more precise in measuring and modeling environmental decoherence. Estimating the present typical uncertainty of typical interferometry experiments as $|\delta\rho/\rho_0| \approx 0.01\%$ we get

$$\lambda_{\mathrm{R}} \gtrsim 10^3, \quad \text{for} \quad C_{70}.$$

We remark that such order of magnitudes estimates are consistent with a small change in the density matrix and the second order Dyson expansion approach.

A value for λ_{R} as small as 10^3 would probably approach the intrinsic spacetime structural limit set by λ, i.e. $\ell = \lambda L_P$. It is interesting to ask what amount of dephasing our model predicts, *independently* of experimental data. To this end we

Figure 4. *Adimensional cutoff parameter as a function of the dephasing rate for a* C_{70} *molecule with a drift time of* 10^{-3} *s.*



need to prescribe theoretically the particle resolution scale L_{R}. Though no obvious choice exists, an interesting possibility would be to set it equal to the particle *Compton length*, i.e.

$$L_{\mathrm{R}} = \frac{h}{Mc}.$$

This choice is obviously Lorentz invariant and also motivated by the fact that the Compton length represents a fundamental uncertainty in the position of a nonrelativistic quantum particle. Indeed, by Heisenberg uncertainty principle, $\Delta x \approx h/Mc$ would imply $\Delta p \gtrsim Mc$, implying an uncertainty in the energy of the same order of the rest mass Mc^2. In such a situation QFT would become relevant. Alternatively it can also be argued that wavelengths shorter than h/Mc would have enough energy to create a particle of mass M from the vacuum. With this choice, equation (36) becomes

$$\left|\frac{\delta\rho}{\rho_0}\right| = \frac{1}{24\pi^3}\left(\frac{M}{M_P}\right)^5\left(\frac{T}{T_P}\right). \tag{37}$$

This can be used to predict the amount of dephasing induced by vacuum conformal fluctuations.

In the case of C_{70} the Compton wavelength is $\approx 10^{-18}\mathrm{m} \approx 10^{18} L_P$, corresponding to $\lambda_{\mathrm{R}} \approx 10^{18}$. For a propagation time of ≈ 1 ms this gives a dephasing rate

$$\left|\frac{\delta\rho}{\rho_0}\right| (M_{C_{70}}, 1\mathrm{ms}) \approx 10^{-44},$$

which would be negligible and far beyond the possibility of experimental detection. Thus, *in order to achieve dephasing rate within the current experimental accuracy, much heavier quantum particles are needed*. In atomic mass units C_{70} has a mass $M_{C_{70}} \approx 10^3$ amu. Equation (37) applied to a particle with mass $M \approx 10^{11}$ amu and with a drift time $T \approx 100$ ms gives the estimate:

$$\left|\frac{\delta\rho}{\rho_0}\right| (10^{11}\mathrm{amu}, 100\mathrm{ms}) \approx 10^{-2}.$$

A drift time of ~ 100 ms could possibly be achieved in a space based experiment. On the other hand, the need of a quantum particle as heavy as 10^{11} amu poses an extraordinary challenge. A possibility would be to employ quantum entangles states. This is already being considered in the literature, e.g. in [29], where entangled atomic states are studied and suggested as a possible improved probe for future detection of spacetime induced dephasing.

The last important point that needs verification is that the condition (4) gave earlier at the beginning of this paper is indeed verified: that was required in order for the change in momentum due to the fluctuations to be smaller than the 'main' particle momentum $\widetilde{p} = M\widetilde{v}$. It read: $\varepsilon^2 \sim \langle A^2 \rangle \ll (\widetilde{v}/c)^2$. The field effective mean quadratic amplitude interacting with the particle is given by $\langle A^2 \rangle \sim {\lambda_{\mathrm{R}}}^{-2}$. Thus we have the condition:

$$\frac{1}{\lambda_{\mathrm{R}}} \equiv \frac{L_P}{L_{\mathrm{R}}} \ll \frac{\widetilde{v}}{c}.$$

By using the expression for the Planck length and with L_{R} given by the particle Compton length, this yields a condition on the particle mass M:

$$\frac{M}{M_{\mathrm{P}}} \ll 2\pi\frac{\widetilde{v}}{c}.$$

For typical laboratory velocities $\widetilde{v}/c \approx 10^{-6}$ and, since $M_{\mathrm{P}} \approx 10^{19}$ amu, this condition is met for particle masses up to $M \approx 10^{13}$ amu, including the case of C_{70} molecules or the heavier entangled quantum states discussed above. This limit would be reduced for slower particles.

We conclude by remarking that the theory described in this paper is quite general, in the sense that as a starting input it only needs a conformally modulated metric and a scalar field satisfying the wave equation. Of course, it is important to identify in concrete which theories of gravity can actually yield such a scenario. This important problem will be the object of future reports, currently in preparation: in [27] a general approach for the study of fluctuating fields close to the Planck scale is introduced and the resulting framework applied to standard GR; while [28] will consider more general scenarios involving scalar-tensor theories of gravity.

Acknowledgments

PB wishes to thank L. Galgani, A. Carati, A. Diaferio and D. Bertacca for the most valuable discussions. Special thanks also go to C. S. Rodrigues, R. Reis, D. Rittich, the University of Aberdeen for a Sixth Century Ph.D. Studentship Award and the STFC Centre for Fundamental Physics.

Appendix A. Stochastic scalar waves and generalized Wiener-Khintchine theorem

Appendix A.1. General solution to the wave equation

In this appendix we work out some general results holding for scalar stochastic waves. Working in units with $c = 1$, the solution to the wave equation $(-\partial_t^2 + \nabla^2) f(\mathbf{x}, t) = 0$ can be written as

$$f(\mathbf{x}, t) = \frac{1}{(2\pi)^n} \int \mathrm{d}^n k \tilde{f}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}},$$

where

$$\tilde{f}(\mathbf{k}, t) = \int \mathrm{d}^n x f(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

and where $f \in L^2(\mathbb{R}^3)$. The Fourier coefficients take the general form

$$\tilde{f}(\mathbf{k}, t) = a(\mathbf{k}) e^{-ikt} + b(\mathbf{k}) e^{ikt}$$

for some *complex functions* $a(\mathbf{k})$ and $b(\mathbf{k})$ and where $k := |\mathbf{k}|$.

The wave can be written as

$$f(\mathbf{x}, t) = \frac{1}{(2\pi)^n} \int \mathrm{d}\hat{\mathbf{k}} \int_0^\infty \mathrm{d}k \, k^{n-1} \left[a(\mathbf{k}) e^{ik(\hat{\mathbf{k}}\cdot\mathbf{x} - t)} + b(-\mathbf{k}) e^{-ik(\hat{\mathbf{k}}\cdot\mathbf{x} - t)} \right],$$

where $\mathrm{d}\hat{\mathbf{k}}$ represent the elementary solid angle in momentum space. Using spherical coordinates in momentum space with $\mathbf{k}(\vartheta, \varphi, k)$ and employing the notation $a_{\hat{\mathbf{k}}}(k) := a(\vartheta, \varphi; k)$ and $b_{-\hat{\mathbf{k}}}(k) := b(\pi - \vartheta, \varphi + \pi; k)$, we define the *directional wave* along $\hat{\mathbf{k}}(\vartheta, \varphi)$ as

$$f_{\hat{\mathbf{k}}}(\hat{\mathbf{k}} \cdot \mathbf{x}/c - t) := \frac{1}{(2\pi)^n} \int_0^\infty \mathrm{d}k \, k^{n-1} \left[a_{\hat{\mathbf{k}}}(k) e^{ikc(\hat{\mathbf{k}}\cdot\mathbf{x}/c - t)} + b_{-\hat{\mathbf{k}}}(k) e^{-ikc(\hat{\mathbf{k}}\cdot\mathbf{x}/c - t)} \right],$$

where for clarity we have restored the speed of light. The general solution can thus be written according to the directional decomposition

$$f(\mathbf{x},t) = \int \mathrm{d}\hat{\mathbf{k}}\, f_{\hat{\mathbf{k}}}(\hat{\mathbf{k}}\cdot\mathbf{x}/c - t). \tag{A.1}$$

Appendix A.2. Stochastic waves

In this section we review some elementary properties of stochastic signals of one time variable t.

The fluctuating conformal field at a given space location $\mathbf{x}$ represents an example of a *stochastic process* $f(t)$. Its properties can be defined in a statistical sense. If the possible values of f obey a probability density $p_t(f)$ the statistical average at time t is defined as

$$\langle f(t)\rangle := \int p_t(f) f df.$$

The *variance* is

$$\left\langle [f(t)]^2 \right\rangle := \int p_t(f) f^2 df.$$

Denoting the probability density of having the particular outcomes $f(t)$ *and* $f(t')$ by $p_{tt'}(f,f')$, then the *autocorrelation* or *2-points correlation function* of $f(t)$ is defined as

$$R(t-t') := \langle f(t)f(t')\rangle := \int p_{tt'}(f,f') f f' df df'.$$

If the probability of having the value $f(t')$ is completely independent from the previous outcome $f(t)$ then $p_{tt'}(f,f') = p_t(f)p_{t'}(f')$ and the stochastic process described by $f(t)$ is said to be perfectly *uncorrelated*. In this case $\langle f(t)f(t')\rangle = \langle f(t)\rangle \langle f(t')\rangle$. Higher order autocorrelation functions can also be defined. The *second order correlation function* is given by

$$R''(t-t') = \left\langle [f(t)]^2[f(t')]^2 \right\rangle := \int p_{tt'}(f,f') f^2 f'^2 df df'.$$

In the case of conformal fluctuations, we assume that the stochastic process is *stationary*, i.e. all average properties do not depend on time, and isotropic. This implies that different directional components have the same statistical properties. Finally, by assuming the stochastic process to be *ergodic*, and since it satisfies the wave equation, then statistical averages can be replaced by time or space averages taken on any given sample function representing the process.

Appendix A.3. Generalized Wiener-Khintchine theorem for a stochastic process satisfying the wave equation

We now derive a generalization of the Wiener-Khintchine theorem linking the power spectrum to the autocorrelation function. We assume that a particular sample function representing the stochastic process can be written as in (A.1), in such a way that the wave equation is satisfied. Some care must be taken in using the Fourier expansions relations of the previous section. Indeed, for a given t, f *is not* in general a square

integrable function belonging to $L^2(\mathbb{R}^3)$. To circumvent this problem, given a sample function of the process and for any given t, we define

$$\tilde{f}^L(\mathbf{k},t) := \int_{\mathcal{D}_L} \mathrm{d}^n x f(\mathbf{x},t) e^{-i\mathbf{k}\cdot\mathbf{x}},$$

where $\mathcal{D}_L$ is a 3-dimensional cubic domain of side L. Then, the function

$$f^L(\mathbf{x},t) := \frac{1}{(2\pi)^n} \int \mathrm{d}^n k \tilde{f}^L(\mathbf{k},t) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{A.2}$$

satisfies the wave equation provided that

$$\tilde{f}^L(\mathbf{k},t) = a^L(\mathbf{k}) e^{-ikt} + b^L(\mathbf{k}) e^{ikt}. \tag{A.3}$$

These expressions will be used later while taking the limit with $L \to \infty$.

Consider now a complex, ergodic, time-dependent stochastic function $f(\mathbf{x},t)$ in an n-dimensional space with time t, satisfying the wave equation $(\partial_t^2 - \nabla^2) f(\mathbf{x},t) = 0$. The autocorrelation function of $f(\mathbf{x},t)$ for any two events $(\mathbf{x}_1,t_1)$ and $(\mathbf{x}_2,t_2) = (\mathbf{x}_1 + \boldsymbol{\xi}, t_1 + \tau)$ is a function of $\boldsymbol{\xi}$ and τ given by

$$C(\boldsymbol{\xi},\tau) = \langle f(\mathbf{x}_1,t_1)^* f(\mathbf{x}_2,t_2) \rangle,$$

and having the property $C(-\boldsymbol{\xi},-\tau) = C(\boldsymbol{\xi},\tau)^*$. For any fixed choice of $\mathbf{x}_1$ and t_1 it also satisfies the wave equation, i.e. $\left(\partial_\tau^2 - \nabla_{\boldsymbol{\xi}}^2\right) C(\boldsymbol{\xi},\tau) = 0$. Assuming that, for any τ, $C(\boldsymbol{\xi},\tau)$ belongs to $L^2(\mathbb{R}^3)$ we have

$$C(\boldsymbol{\xi},\tau) = \frac{1}{(2\pi)^n} \int \mathrm{d}^n k \left[\alpha(\mathbf{k}) e^{-ik\tau} + \beta(\mathbf{k}) e^{ik\tau}\right] e^{i\mathbf{k}\cdot\boldsymbol{\xi}}. \tag{A.4}$$

Notice that $\alpha(\mathbf{k})^* = \alpha(\mathbf{k}), \quad \beta(\mathbf{k})^* = \beta(\mathbf{k})$, i.e. both $\alpha(\mathbf{k})$ and $\beta(\mathbf{k})$ are real. For $\tau = 0$ (A.4) reduces to the case of stochastic process of one n-dimensional space variable:

$$C(\boldsymbol{\xi},0) = \frac{1}{(2\pi)^n} \int \mathrm{d}^n k \left[\alpha(\mathbf{k}) + \beta(\mathbf{k})\right] e^{i\mathbf{k}\cdot\boldsymbol{\xi}}.$$

The standard W-K theorem then implies a power spectrum:

$$S(\mathbf{k}) = \alpha(\mathbf{k}) + \beta(\mathbf{k}). \tag{A.5}$$

To determine $\alpha(\mathbf{k})$ and $\beta(\mathbf{k})$ we consider the stochastic process for some fixed time t_0. From equations (A.2) and (A.3) we have

$$f^L(\mathbf{x},t_0) = \frac{1}{(2\pi)^n} \int \mathrm{d}^n k \tilde{f}^L(\mathbf{k},t_0) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{A.6}$$

with

$$\tilde{f}^L(\mathbf{k},t_0) = a^L(\mathbf{k}) e^{-ikt_0} + b^L(\mathbf{k}) e^{ikt_0}. \tag{A.7}$$

We are thus dealing with one space variable stochastic process for which the usual W-K theorem holds. Exploiting the fact that the stochastic process is stationary we define mean power spectral density as

$$S(\mathbf{k}) := \lim_{L,T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t_0 \, \frac{1}{L^n} \left\langle \tilde{f}^L(\mathbf{k},t_0)^* \tilde{f}^L(\mathbf{k},t_0) \right\rangle.$$

Using equations (A.6) and (A.7) this reduces to

$$S(\mathbf{k}) = \lim_{L\to\infty} \frac{1}{L^n} \left\langle \left|a^L(\mathbf{k})\right|^2 + \left|b^L(\mathbf{k})\right|^2 \right\rangle.$$

Comparing with (A.5) we have

$$S(\mathbf{k}) = \lim_{L\to\infty} \frac{1}{L^n} \left\langle \left|a^L(\mathbf{k})\right|^2 + \left|b^L(\mathbf{k})\right|^2 \right\rangle.$$

Appendix A.3.1. Real stochastic process If $f(\mathbf{x},t)$ is real it is easily verified that $S(\mathbf{k})$ *is even*, i.e. $S(\mathbf{k}) = S(-\mathbf{k})$ and that $C(\boldsymbol{\xi},\tau)$ *is real*. Moreover the power spectrum is then given simply by

$$S(\mathbf{k}) = \lim_{L\to\infty} \frac{2}{L^n} \left\langle \left| a^L(\mathbf{k}) \right|^2 \right\rangle.$$

In this case we get the generalized Wiener-Khintchine theorem for a real, stationary stochastic scalar in the form:

$$C(\boldsymbol{\xi},\tau) = \frac{1}{(2\pi)^n} \int \mathrm{d}^n k\, S(\mathbf{k}) \cos(\mathbf{k}\cdot\boldsymbol{\xi} - kc\tau),$$

where we have restored the speed of light.

Appendix A.4. Correlation properties of wave components in different directions

The results that we have come to establish allow to show that wave components traveling in different directions are uncorrelated. Evaluating $\langle f^*(\mathbf{x}_1,t_1) f(\mathbf{x}_2,t_2)\rangle$ using equation (A.1) we have

$$C(\boldsymbol{\xi},\tau) = \langle f^*(\mathbf{x},t) f(\mathbf{x}+\boldsymbol{\xi}, t+\tau)\rangle = \int \mathrm{d}\hat{\mathbf{k}} \int \mathrm{d}\hat{\mathbf{k}}' \left\langle f^*_{\hat{\mathbf{k}}}(\hat{\mathbf{k}}\cdot\mathbf{x}/c - t)\, f_{\hat{\mathbf{k}}'}([\hat{\mathbf{k}}'\cdot\mathbf{x}/c - t] + [\hat{\mathbf{k}}'\cdot\boldsymbol{\xi}/c - \tau]) \right\rangle. \tag{A.8}$$

Using the Wiener-Khintchine theorem in its form as given by (A.4), restoring the speed of light, swapping $\mathbf{k}$ to $-\mathbf{k}$ in the second integral and splitting the $\mathrm{d}^n k$ integral in its angular and magnitude parts we can write as well

$$C(\boldsymbol{\xi},\tau) = \int \mathrm{d}\hat{\mathbf{k}}\, C_{\hat{\mathbf{k}}}(\hat{\mathbf{k}}\cdot\boldsymbol{\xi}/c - \tau), \tag{A.9}$$

where we defined the correlation function in the direction $\hat{\mathbf{k}}$ as

$$C_{\hat{\mathbf{k}}}(\hat{\mathbf{k}}\cdot\boldsymbol{\xi}/c - \tau) := \frac{1}{(2\pi)^n} \int_0^\infty \mathrm{d}k\, k^{n-1} [\alpha_{\hat{\mathbf{k}}}(k) e^{ikc(\hat{\mathbf{k}}\cdot\boldsymbol{\xi}/c-\tau)} + \beta_{-\hat{\mathbf{k}}}(k) e^{-ikc(\hat{\mathbf{k}}\cdot\boldsymbol{\xi}/c-\tau)}],$$

with $\alpha_{\hat{\mathbf{k}}}(k) := \alpha(\mathbf{k})$ and $\beta_{-\hat{\mathbf{k}}}(k) := \beta(-\mathbf{k})$. The two equations (A.8) and (A.9) must be equivalent. This implies at once the following equation

$$\left\langle f^*_{\hat{\mathbf{k}}}(\hat{\mathbf{k}}\cdot\mathbf{x}/c - t)\, f_{\hat{\mathbf{k}}'}([\hat{\mathbf{k}}'\cdot\mathbf{x}/c - t] + [\hat{\mathbf{k}}'\cdot\boldsymbol{\xi}/c - \tau]) \right\rangle = \delta(\mathbf{k},\mathbf{k}')\, C_{\hat{\mathbf{k}}}(\hat{\mathbf{k}}\cdot\boldsymbol{\xi}/c - \tau)$$

or, equivalently, since $\hat{\mathbf{k}}\cdot\mathbf{x}/c$ has the dimensions of a time,

$$\left\langle f^*_{\hat{\mathbf{k}}}(t)\, f_{\hat{\mathbf{k}}'}(t+\tau) \right\rangle = \delta(\mathbf{k},\mathbf{k}')\, C_{\hat{\mathbf{k}}}(\tau). \tag{A.10}$$

We thus see that the fluctuating field can be resolved into components along different directions represented by completely uncorrelated functions of *just one time variable*.

Appendix A.5. Isotropic power spectrum and field averages

In the relevant case of a real and isotropic signal in 3-dimensional space the spectral density must also be isotropic, i.e. $S(\mathbf{k}) := S(k)$. We can then introduce an *isotropic correlation function* as

$$C(\tau) := \frac{1}{(2\pi)^3} \int_0^\infty \mathrm{d}k\, k^2 S(k) \cos(kc\tau) \tag{A.11}$$

and (A.10) reads simply $\left\langle f_{\hat{\mathbf{k}}}(t)\, f_{\hat{\mathbf{k}}'}(t+\tau)\right\rangle = \delta(\mathbf{k},\mathbf{k}')\, C(\tau)$. Using this together with (A.1), we have

$$\left\langle f(\mathbf{x},t)^2\right\rangle = \left\langle \int \mathrm{d}\hat{\mathbf{k}} f_{\hat{\mathbf{k}}}(\mathbf{x},t) \int \mathrm{d}\hat{\mathbf{k}}' f_{\hat{\mathbf{k}}'}(\mathbf{x},t)\right\rangle = C_0 \int \mathrm{d}\hat{\mathbf{k}} \int \mathrm{d}\hat{\mathbf{k}}' \delta(\mathbf{k},\mathbf{k}'),$$

so that

$$\left\langle f(\mathbf{x},t)^2\right\rangle = 4\pi C_0.$$

Using this with equation (A.11) we have

$$\left\langle f(\mathbf{x},t)^2\right\rangle = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, S(k).$$

It is also useful to deduce two expressions for the average of the square of the time and space derivatives of the field, as these are directly related to the field energy density. Using equation (A.2) and the ergodic property it is straightforward to show that

$$\left\langle (\partial_t f)^2\right\rangle = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, k^2 S(\mathbf{k}),$$

and

$$\left\langle |\nabla f|^2\right\rangle = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, k^2 S(\mathbf{k}).$$

Appendix B. Treatment of the term T_4

We consider the correction term $T_4 = -i\hbar\left(\dot{A} - A\dot{A}\right)\psi - i\hbar\widetilde{v}\left(A_{,x} - AA_{,x}\right)\psi$, derived in section 2. Writing A, in the isotropic and real case, as:

$$A \approx \int \mathrm{d}\hat{\mathbf{k}} \int_0^{2\pi/L_{\mathrm{R}}} \mathrm{d}k\, k^2 a(k) e^{ik(\hat{\mathbf{k}}\cdot\mathbf{x}-ct)}, \tag{B.1}$$

where the upper cutoff is set by the particle resolution scale L_{R}. The power spectrum is basically proportional to the square of the Fourier component $a(k)$. For $S(k)\sim 1/k$ we have $a(k)\sim k^{-1/2}$, so that the effective coefficient appearing in the expansion is $k^2 a(k)\sim k^{3/2}$. Thus the short wavelengths close to the cutoff give the most important contribution. For this reason we can approximate the field as

$$A \approx \int \mathrm{d}\hat{\mathbf{k}} \int_0^{2\pi/L_{\mathrm{R}}} \mathrm{d}k\, k^{3/2} \Delta(k-k_A) e^{ik(\hat{\mathbf{k}}\cdot\mathbf{x}-ct)}, \tag{B.2}$$

where the function $\Delta(k-k_A)$ is peaked around a typical wave number k_A. This can in principle be selected in such a way that the average properties of (B.1) are equivalent to those of (B.2). Effectively we get:

$$A \approx \int \mathrm{d}\hat{\mathbf{k}} {k_A}^{3/2} e^{ik_A(\hat{\mathbf{k}}\cdot\mathbf{x}-ct)}, \tag{B.3}$$

so that the conformal field is approximated as a fast varying and isotropic random signal characterized by a *single typical wavelength* $\lambda_A \equiv 2\pi/k_A$. In relation to the fluctuations ability to affect the particle, this will be close to the particle resolution scale, i.e. we put $\lambda_A \equiv \kappa L_{\mathrm{R}}$, with $\kappa \gtrsim 1$. From (B.3) we now have:

$$\dot{A} \approx -ik_A c A = -\frac{2\pi i}{\kappa L_{\mathrm{R}}} cA = -\frac{iMc^2}{\kappa\hbar} A,$$

where we used $L_{\mathrm{R}} = h/Mc$. The space derivatives yield:

$$A_{,x} \approx i{k_A}^{5/2}\int \mathrm{d}\hat{\mathbf{k}}\hat{k}_x e^{ik_A(\hat{\mathbf{k}}\cdot\mathbf{x}-ct)} = 0.$$

Using these two relations in (6) yields the result (7).

Appendix C. An integral identity

In this appendix we prove that the result

$$I := \int_0^T \mathrm{d}t \left[2 \int_0^t \mathrm{d}t' f(t-t') - \int_0^T \mathrm{d}t' f(t-t'-\hat{\mathbf{k}}\cdot\Delta\mathbf{x}) \right] = 0$$

holds for an *arbitrary even function* f and in the limit $\hat{\mathbf{k}}\cdot\Delta\mathbf{x}/T \to 0$.

For simplicity let $\Delta := \hat{\mathbf{k}}\cdot\Delta\mathbf{x}$. Defining the variable $\tau := t - t'$ we have

$$\int_0^t \mathrm{d}t' f(t-t') = \int_0^t \mathrm{d}\tau f(\tau),$$

while, with $\tau := t - t' - \Delta$,

$$\int_0^T \mathrm{d}t' f(t-t'-\Delta) = \int_{t-\Delta-T}^{t-\Delta} \mathrm{d}\tau f(\tau) = \int_0^{t-\Delta} \mathrm{d}\tau f(\tau) - \int_0^{t-\Delta-T} \mathrm{d}\tau f(\tau).$$

Introducing the primitive of $f(t)$

$$F(t) := \int_0^t \mathrm{d}\tau f(\tau)$$

we can re-write I as

$$I = 2\int_0^T \mathrm{d}t F(t) - \int_0^T \mathrm{d}t\, F(t-\Delta) + \int_0^T \mathrm{d}t\, F(t-\Delta-T).$$

Performing a further change of variable $z := t - \Delta$ the second integral reads

$$\int_0^T \mathrm{d}t\, F(t-\Delta) = \int_{-\Delta}^{T-\Delta} dz\, F(z) := \int_0^{T-\Delta} dz\, F(z) - \int_0^{-\Delta} dz\, F(z).$$

Performing a similar operation on the third integral we obtain

$$I = 2\int_0^T \mathrm{d}t F(t) - \int_0^{T-\Delta} dz\, F(z) + \int_0^{-\Delta} dz\, F(z) + \int_0^{-\Delta} dz\, F(z) - \int_0^{-T-\Delta} dz\, F(z).$$

Now we use the information $T \gg \Delta$ and $f(t) = f(-t)$. As an elementary consequence we have that $F(t)$ is $F(t) = -F(-t)$, implying that

$$\int_0^{\chi} \mathrm{d}\tau F(\tau) = \int_0^{-\chi} \mathrm{d}\tau F(\tau)$$

Approximating $T \pm \Delta \approx T$ and swapping the sign of the upper integration bound appropriately we have

$$I = 2\int_0^T \mathrm{d}t F(t) - \left[\int_0^T dz\, F(z) - \int_0^{\Delta} dz\, F(z) \right] - \left[\int_0^T dz\, F(z) - \int_0^{\Delta} dz\, F(z) \right].$$

The integrals from 0 to Δ can all be neglected exploiting again the fact that $T \gg \Delta$ and we obtain

$$I \approx 2\int_0^T \mathrm{d}t F(t) - 2\int_0^T \mathrm{d}t F(t) = 0.$$

The result in exact in the limit $\Delta/T \to 0$.

Appendix D. Fourth order term in the Dyson expansion

The fourth order propagator in the Dyson expansion is

$$\hat{K}_4(T) := \left(\frac{-i}{\hbar}\right)^4 \int_0^T \mathrm{d}t^{(1)} \int_0^{t^{(1)}} \mathrm{d}t^{(2)} \int_0^{t^{(2)}} \mathrm{d}t^{(3)} \int_0^{t^{(3)}} \mathrm{d}t^{(4)} \hat{H}(t^{(1)})\hat{H}(t^{(2)})\hat{H}(t^{(3)})\hat{H}(t^{(4)}), \tag{D.1}$$

implying a fourth order term in the expression for the density matrix evolution given by

$$\left\langle \hat{K}_4\rho_0 \right\rangle \sim \int_0^T \mathrm{d}t^{(1)} \int_0^{t^{(1)}} \mathrm{d}t^{(2)} \int_0^{t^{(2)}} \mathrm{d}t^{(3)} \int_0^{t^{(3)}} \mathrm{d}t^{(4)} \int \mathrm{d}^3y^{(1)} |\mathbf{y}^{(1)}\rangle\langle \mathbf{y}^{(1)}| \cdots \int \mathrm{d}^3y^{(4)} |\mathbf{y}^{(4)}\rangle\langle \mathbf{y}^{(4)}| \times$$
$$\times \left\langle V(\mathbf{y}^{(1)}, t^{(1)}) V(\mathbf{y}^{(2)}, t^{(2)}) V(\mathbf{y}^{(3)}, t^{(3)}) V(\mathbf{y}^{(4)}, t^{(4)}) \right\rangle \rho_0.$$

Since the potential is $V = Mc^2(C_1A + C_2A^2)$ the average yields one term proportional to A_0^4:

$$\left\langle V(\mathbf{y}^{(1)}, t^{(1)}) V(\mathbf{y}^{(2)}, t^{(2)}) V(\mathbf{y}^{(3)}, t^{(3)}) V(\mathbf{y}^{(4)}, t^{(4)}) \right\rangle \Rightarrow$$
$$\left(\frac{Mc^2 A_0}{\hbar}\right)^4 \int \mathrm{d}\hat{\mathbf{k}}^{(1)} \cdots \int \mathrm{d}\hat{\mathbf{k}}^{(4)} \left\langle f_{\hat{\mathbf{k}}}(\tau^{(1)}) f_{\hat{\mathbf{k}}'}(\tau^{(2)}) f_{\hat{\mathbf{k}}}(\tau^{(3)}) f_{\hat{\mathbf{k}}'}(\tau^{(4)}) \right\rangle,$$

where $\tau^{(i)} := t^{(i)} - \mathbf{y}^{(i)} \cdot \hat{\mathbf{k}}^{(i)}$. This requires knowledge of the 4-point correlation function, involving the average of the product of 4 directional components evaluated at different points. For a real random process having a zero mean and gaussian distribution the 4-points function reduces to [38, 39]:

$$\left\langle f_{\hat{\mathbf{k}}}(\tau^{(1)}) f_{\hat{\mathbf{k}}'}(\tau^{(2)}) f_{\hat{\mathbf{k}}}(\tau^{(3)}) f_{\hat{\mathbf{k}}'}(\tau^{(4)}) \right\rangle = \left\langle f_{\hat{\mathbf{k}}}(\tau^{(1)}) f_{\hat{\mathbf{k}}'}(\tau^{(2)}) \right\rangle \left\langle f_{\hat{\mathbf{k}}}(\tau^{(3)}) f_{\hat{\mathbf{k}}'}(\tau^{(4)}) \right\rangle$$
$$+ \left\langle f_{\hat{\mathbf{k}}}(\tau^{(1)}) f_{\hat{\mathbf{k}}'}(\tau^{(3)}) \right\rangle \left\langle f_{\hat{\mathbf{k}}}(\tau^{(2)}) f_{\hat{\mathbf{k}}'}(\tau^{(4)}) \right\rangle$$
$$+ \left\langle f_{\hat{\mathbf{k}}}(\tau^{(1)}) f_{\hat{\mathbf{k}}'}(\tau^{(4)}) \right\rangle \left\langle f_{\hat{\mathbf{k}}}(\tau^{(2)}) f_{\hat{\mathbf{k}}'}(\tau^{(3)}) \right\rangle.$$

We can now use equation (14) to express the 2-point correlations:

$$\left\langle f_{\hat{\mathbf{k}}}(\tau^{(1)}) f_{\hat{\mathbf{k}}'}(\tau^{(2)}) f_{\hat{\mathbf{k}}}(\tau^{(3)}) f_{\hat{\mathbf{k}}'}(\tau^{(4)}) \right\rangle =$$
$$\delta(\hat{\mathbf{k}}^{(1)}, \hat{\mathbf{k}}^{(2)}) R(\tau^{(1)} - \tau^{(2)}) \delta(\hat{\mathbf{k}}^{(3)}, \hat{\mathbf{k}}^{(4)}) R(\tau^{(3)} - \tau^{(4)})$$
$$+\delta(\hat{\mathbf{k}}^{(1)}, \hat{\mathbf{k}}^{(3)}) R(\tau^{(1)} - \tau^{(3)}) \delta(\hat{\mathbf{k}}^{(2)}, \hat{\mathbf{k}}^{(4)}) R(\tau^{(2)} - \tau^{(4)})$$
$$+\delta(\hat{\mathbf{k}}^{(1)}, \hat{\mathbf{k}}^{(4)}) R(\tau^{(1)} - \tau^{(4)}) \delta(\hat{\mathbf{k}}^{(2)}, \hat{\mathbf{k}}^{(3)}) R(\tau^{(2)} - \tau^{(3)}).$$

This implies that the term $T_4[A_0^4]$ deriving from $\left\langle \hat{K}_4\rho_0 \right\rangle$ has the structure:

$$T_4[A_0^4] \sim 3 \left[\int \mathrm{d}\hat{\mathbf{k}}^{(1)} \int \mathrm{d}\hat{\mathbf{k}}^{(2)} \int_0^T \mathrm{d}t^{(1)} \int_0^T \mathrm{d}t^{(2)} \delta(\hat{\mathbf{k}}^{(1)}, \hat{\mathbf{k}}^{(2)}) R(\tau^{(1)} - \tau^{(2)}) \right]^2,$$

where all upper bounds in the time integration can be set equal to T by appropriate normal ordering [36]. By carrying out one of the two angular integrations we have:

$$T_4[A_0^4] \sim 3 \left[\int \mathrm{d}\hat{\mathbf{k}}^{(1)} \int_0^T \mathrm{d}t^{(1)} \int_0^T \mathrm{d}t^{(2)} R[t^{(1)} - t^{(2)} - \hat{\mathbf{k}}^{(1)} \cdot (\mathbf{y}^{(1)} - \mathbf{y}^{(2)})] \right]^2.$$

The double time integral can be simplified using the general result (21) and we have

$$T_4[A_0^4] \sim 3 \left[\int \mathrm{d}\hat{\mathbf{k}}^{(1)} \mathfrak{F} \left[R(t + \tau) \right](0) T \right]^2,$$

where $\mathfrak{F}$ denotes Fourier transform and $\tau := -\hat{\mathbf{k}}^{(1)}\cdot(\mathbf{y}^{(1)}-\mathbf{y}^{(2)})$. The Fourier transform can be evaluated using that $R(t+\tau) = C(t+\tau)/C_0$. Then

$$\begin{aligned}\mathfrak{F}\left[R(t+\tau)\right](\omega) &= \frac{1}{C_0}\int_{-\infty}^{\infty} \mathrm{d}t C(t+\tau)e^{-i\omega t}\\ &= \frac{1}{C_0(2\pi c)^3}\int_0^{\omega_c}\mathrm{d}\omega'\int_{-\infty}^{\infty}\mathrm{d}t\,\omega'^2 S(\omega')e^{-i\omega t}\cos\omega'(t+\tau)\\ &= \frac{1}{2C_0(2\pi c)^3}\int_0^{\omega_c}\mathrm{d}\omega'\int_{-\infty}^{\infty}\mathrm{d}t\,\omega'^2 S(\omega')e^{-i\omega t}\left[e^{i\omega'(t+\tau)}+e^{-i\omega'(t+\tau)}\right]\\ &= \frac{1}{2C_0(2\pi c)^3}\int_0^{\omega_c}\mathrm{d}\omega'\omega'^2 S(\omega')\int_{-\infty}^{\infty}\mathrm{d}t\,\left[e^{i(\omega'-\omega)t}e^{i\omega'\tau}+e^{i(-\omega'-\omega)t}e^{-i\omega'\tau}\right].\end{aligned}$$

Integrating with respect to t gives

$$\mathfrak{F}\left[R(t+\tau)\right](\omega) = \frac{1}{2C_0(2\pi c)^3}\int_0^{\omega_c}\mathrm{d}\omega'\omega'^2 S(\omega')\left[\delta(\omega'-\omega)e^{i\omega'\tau}+\delta(-\omega'-\omega)e^{-i\omega'\tau}\right].$$

This vanishes for $\omega > \omega_c$. For $\omega = 0$ we have:

$$\mathfrak{F}\left[R(t+\tau)\right](0) = \frac{1}{2C_0(2\pi c)^3}\int_0^{\omega_c}\mathrm{d}\omega'\omega'^2 S(\omega')\left[\delta(\omega')e^{i\omega'\tau}+\delta(-\omega')e^{-i\omega'\tau}\right].$$

Carrying out the frequency integral and using the properties of the δ function we have:

$$\mathfrak{F}\left[R(t+\tau)\right](0) = \frac{1}{C_0(2\pi c)^3}\lim_{\omega\to 0}\omega^2 S(\omega).$$

In the interesting case $S \sim 1/\omega$ this tends to 0, in such a way that $T_4[A_0^4] \to 0$ and the fourth order Dyson expansion term doesn't give any contribution.

References

[1] Power W L and Percival I C 2000 Decoherence of quantum wave packets due to interaction with conformal space-time fluctuations *Proc. R. Soc. Lond. A* **456** 955

[2] Bingham R 2005 The search for quantum gravity using matter interferometers *Phys. Scr.* **T116** 132-34

[3] Wang C H–T, Bingham R and Mendonça J T 2006 Quantum gravitational decoherence of matter waves *Class. Quantum Grav.* **23** L59-L65

[4] Amelino-Camelia G, Lämmerzahl C, Macias M and Müller H 2005 The search for quantum gravity signals *Preprint* gr–qc/0501053v1

[5] Anastopoulos C and Hu B L 2007 Decoherence in quantum gravity: issues and critiques *J. Phys. Conf. Ser.* **67** 012012 (*Preprint* gr–qc/0703137v1)

[6] Breuer H–P, Göklü E and Lämmerzahl C 2008 Metric fluctuations and decoherence *Preprint* gr–qc/0812.0420

[7] Hornberger K and Sipe J E 2003 Collisional decoherence reexamined *Phys. Rev. A* **68** 012105

[8] Hackermüller L, Uttenthaler S, Hornberger K, Reiger E, Brezger B, Zeilinger A and Arndt M 2003 Wave nature of biomolecules and fluorofullerenes *Phys. Rev. Lett.* **91** 9

[9] Hornberger K, Sipe J E and Arndt M 2004 Theory of decoherence in a matter wave Talbot-Lau interferometer *Phys. Rev. A* **70** 053608

[10] Hackermüller L, Hornberger K, Brezger B, Zeilinger A and Arndt M 2004 Decoherence of matter waves by thermal emission of radiation *Nature* **427** 711

[11] Stibor A, Hornberger K, Hackermüller L, Zeilinger A and Arndt M 2005 *Laser Physics* **15**(1) 10-17

[12] Reynaud S, Lamine B, Lambrecht A, Neto P M and Jaekel M-T 2004 HYPER and gravitational decoherence *General Relativity and Gravitation* **36** 10

[13] Kiefer C 2005 Quantum gravity: general introduction and recent developments *Ann. Phys.* **15** 129–48

[14] Boyer T H 1969 Derivation of the blackbody radiation spectrum withut quantum assumptions *Phys. Rev.* **182**(5) 1374–83

[15] Boyer T H 1975 Random electrodynamics: the theory of classical electrodynamics with classical electromagnetic zero point radiation, *Phys. Rev. D* **11**(4) 790
[16] Frederick C 1976 Stochastic spacetime and quantum theory *Phys. Rev. D* **13**(12) 3183
[17] Moffat J W 1997 Stochastic gravity *Phys. Rev. D* **56**(10) 6264
[18] Miller S D 1999 Einstein-Langevin and Einstein-Fokker-Planck equations for Oppenheimer-Snyder gravitational collapse in a spacetime with conformal vacuum fluctuations *Class. Quantum Grav.* **16** 33813403
[19] Göklü E and Lämmerzahl C 2008 Metric fluctuations and the weak equivalence principle *Class. Quantum Grav.* **25** 105012
[20] Kastrup H A 2008 On the advancements of conformal transformations and their associated symmetries in geometry and theoretical physics *Ann. Phys.* **17** 631 (*Preprint* physics.hist-ph/0808.2730v1)
[21] Nelson E 1985 *Quantum Fluctuations* (Princeton, NJ: Princeton University Press)
[22] Dicke R H 1962 Mach's principle and invariance under transformation of units *Phys. Rev.* **125** 6 2163
[23] Wagoner R V 1970 Scalar tensor theory and gravitational wave *Phys. Rev. D* **1** 12 3209
[24] Stern A, Aharonov Y and Imry Y 1990 Phase uncertainty and loss of interference *Phys. Rev. A* **41** 7 3436
[25] Wang C H–T 2005 Conformal geometrodynamics: true degrees of freedom in a truly canonical structure *Phys. Rev. D* **71**, 124026
[26] Wang C H–T 2005 Unambiguous spin-gauge formulation of canonical general relativity with conformorphism invariance *Phys. Rev. D* **72** 087501
[27] Bonifacio P M 2009 The nonlinear random gravity approach to the problem of spacetime fluctuations. I. GR and conformal fluctuations, *In preparation*
[28] Bonifacio P M and Wang C H–T 2009 The nonlinear random gravity approach to the problem of spacetime fluctuations. II. scalar tensor theories and conformal fluctuations *In preparation*
[29] Everitt M S, Jones M L and Varcoe B T H 2008 Dephasing of entangled atoms as an improved test of quantum gravity *Preprint* gr–qc/0812.3052v2
[30] DeWitt B S 1966 *Phys. Rev. Lett.* **16** 1092
[31] Linet B and Tourrenc P 1976 Changement de phase dans un champ de gravitation: possibilité de détection interférentielle *Can. J. Phys.* **54** 1129
[32] Cai Y Q and Papini G 1989 Particle interferometry in weak gravitational fields *Class. Quantum Grav.* **6** 407–18
[33] Kiefer C and Singh T P 1991 Quantum gravitational corrections to the functional Schrödinger equation *Phys. Rev. D* **44**(4) 1067
[34] Wald R M 1984 *General Relativity* The University of Chicago press
[35] Roura A, Brill D R, Hu B L, Misner C W, and Phillips W D, 2006 *Gravitational wave detectors based on matter wave interferometers (MIGO) are no better than laser interferometers (LIGO) Phys. Rev. D* **73** 084018
[36] Roman P 1965 *Advanced Quantum Theory; an Outline of the Fundamental Ideas* Addison-Wesley Pub. Co.
[37] Ryder L H 1985 *Quantum Field Theory* Cambridge University Press
[38] Adler R 1981 *The Geometry of Random Fields* John Wiley & Sons.
[39] Eriksen H K, Banday A J and Górski K M 2002 The N-point Correlation Functions of the COBE-DMR Maps Revisited *Astron. Astrophys.* **395** 409–416